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Geometric quantization

G.M. Tuynman

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PREFACE

This manuscript contains the notes of a series of lectures I gave at the seminar "mathematical structures in field theories" during the academic years 1983-1984 and 1984-1985. They are meant as an introduction to the theory known as geometric quantization and although it does not cover as many aspects as other books on geometric quantization, I have tried to give full proofs where other authors only sketch them or leave them out altogether. This explains part of the origin of these notes: to satisfy myself that I could prove the stated claims, the other part being the request to supply the audience of the seminar with a written text. It is assumed that the reader is familiar with the fundamentals of differential geometry, (fibre) bundle theory, symplectic geometry, hamiltonian mechanics and quantum mechanics.

At this point I would like to thank the organizers E.M. de Jager and H.G.J. Pijls for giving me the opportunity to talk at their seminar. I also would like to thank mw Y.Voorn for the care with which she has typed this manuscript, and finally I thank the CWI for publishing it in their series CWI-syllabi.

G.M. Tuynman
August 1985

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1 INTRODUCTION

Suppose we are given a physical system which we can describe in classical mechanics (e.g. a single particle in $\mathbb{R}^3$ in a potential field), then we can ask ourselves: how do we describe this system in quantum mechanics? This question implies that we have to find a Hilbert space H and, for each (classical) observable, a self-adjoint operator on H (especially for the hamilton function which determines the time evolution of the system). In theory one is completely free in the choice of H and the self-adjoint operators, provided one is in agreement with experiment, but in practice one uses the description by classical mechanics as a guide to construct H and the self-adjoint operators. This practical method to determine the quantum mechanical description immediately evokes the question:

how does classical mechanics guide us?

In the history of quantum mechanics several remarks have been made concerning this question, some of which will be summed up below.
In his book "The principles of Quantum Mechanics" Dirac noticed a striking resemblance between Poisson-brackets in classical mechanics and the commutator of operators in quantum mechanics when applied to the observables position (q^i), momentum (p_i) and energy H. This resemblance between Poisson brackets and commutator seemed good enough to be promoted to the so-called canonical quantization procedure in which one tries to represent the canonically conjugated observables q^i and p_i by self-adjoint operators Q^i, P_i satisfying:

$$[Q^i,Q^j] = 0 = [P_i,P_j], \quad [P_i,Q^j] = -i\hbar\delta_i^j = -i\hbar\,[p_i,q^j]$$

Furthermore, functions of the q^i (or p_i) are in this scheme represented by the same functions of Q^i (resp. P_i). However, the canonical quantization procedure does not give a prescription for observables which are functions of p_i and q^i simultaneously, in particular, it does not give a prescription how to "quantize" the observable $p_1 q^1$ (classically $p_1 q^1$ is the same as $q^1 p_1$, but quantum mechanically P_1 and Q^1 do not commute!).

An alternative approach was given by Weyl [Weyl 1927] who proposed a procedure to quantize observables using the Fourier transform. This procedure coincides with the canonical quantization procedure for the elementary observables; however, for more complicated observables (which "cannot" be quantized by the canonical quantization procedure) the correspondence between Poisson brackets and commutator is not preserved.

In the same time Stone and von Neumann proved that the Schrödinger representation is unique. That is to say, given a set of self-adjoint operators P_i and Q^i $(i = 1,..,n)$ on a Hilbert space $\mathcal{H}$ such that their associated one-parameter unitary groups $\exp(i P_j t)$, $\exp(i Q^j t)$ satisfy:

(i) $\exp(i P_j t) \exp(i P_k s) = \exp(i P_k s) \exp(i P_j t)$
$\exp(i Q^j t) \exp(i Q^k s) = \exp(i Q^k s) \exp(i Q^j t)$
$\exp(i P_j t) \exp(i Q^k s) = \exp(i\, st\, \delta^k_j) \exp(i Q^k s) \exp(i P_j t)$

(ii) $\mathcal{H}$ is irreducible under the action of $\exp(i P_j t)$ and $\exp(i Q^j t)$.

Then there exist a unitary map $S: \mathcal{H} \to L^2(\mathbb{R}^n)$ ($L^2(\mathbb{R}^n)$ = complex square-integrable functions on $\mathbb{R}^n$ with respect to the Lebesgue measure) such that for f in a suitable dense subset of $L^2(\mathbb{R}^n)$:

$$(SP_j S^{-1} f)(q) = i\hbar \frac{\partial f}{\partial q^j}(q) \quad , \quad (SQ^j S^{-1} f)(q) = q^j f(q) .$$

The relations (i) are called the Weyl (commutation) relations and from these relations one can deduce that P_j and Q^k satisfy

$$[P_i, P_j] = 0 = [Q^i, Q^j], \quad [P_i, Q^j] = -i\hbar\, \delta^j_i$$

which are the usual commutation relations of the canonically conjugated observables p_i and q^j of the canonical quantization procedure. However, it should be noted that in general one cannot deduce the Weyl commutation relations from the commutation relations of P_i and Q^j (see [Reed & Simon]). When we omit condition (ii), $\mathcal{H}$ becomes the direct sum of (countably many) irreducible parts which are all equivalent to the Schrödinger representation ($P_i \leftrightarrow -i\hbar\frac{\partial}{\partial q^i}$, $Q^i \leftrightarrow q^i$ on $L^2(\mathbb{R}^n)$).

Given the succes of the Schrödinger representation in quantum mechanics one now would like to use this theorem "in reverse" to formalize the canonical quantization procedure, i.e. to obtain a set of rules which describes the quantum mechanical ingredients in terms of the classical ones. In view of the previous remarks, the following conditions seem reasonable: for a classical system described by canonically conjugated coordinates (q^i,p_j) and a set of (classical) observables C such that $q^i,p_j \in C$, the quantum description is given by a Hilbert space $\mathcal{H}$ and a map $\delta: C \to$ {self-adjoint operators on $\mathcal{H}$} such that:

(Qi) $\delta(\phi+\psi) = \delta(\phi) + \delta(\psi)$

(Qii) $\delta(\lambda\phi) = \lambda\delta(\phi)$, $\lambda \in \mathbb{R}$

(Qiii) $\delta([\phi,\psi]) = -i\hbar[\delta(\phi), \delta(\psi)]$

(Qiv) $\delta(1) = \mathbb{1}$

(Qv) $\mathcal{H}$ is irreducible under the action of $\delta(q^i)$ and $\delta(p_j)$

Conditions (Qi)-(Qiv) say that δ is a 1-1 representation of the Poisson algebra C to the Lie algebra of quantum observables. Condition (Qv) can be interpreted as follows: the irreducibility of $\mathcal{H}$ under $\delta(q^i)$ and $\delta(p_j)$ implies that no proper subspace of $\mathcal{H}$ is invariant under the action of $\delta(q^i)$ and $\delta(p_j)$ which is heuristically equivalent to its classical counterpart: no subspace of the phase-space (q,p) is invariant under all translations $q^i \to q^i + a^i$, $p_j \to p_j + b_j$; the latter condition, being an "obvious truth" of classical mechanics, justifies condition (Qv).

Now Van Hove proved that these 5 conditions are too strong: there cannot exist a Hilbert space H and a map δ satisfying all requirements simultaneously [Van Hove 1951]. In the same paper he showed that if one drops requirement (Qv) *or* if one represents only a restricted class of observables, then a H and δ can exist. (This result of Van Hove has been generalized by Gotay [Gotay 1980] who uses Van Hove's result to prove that a quantization procedure satisfying a certain set of "obvious" conditions (similar to our 5) cannot exist.)

Considering these results we may conclude that the original striking resemblance between Poisson brackets and commutators remains a striking resemblance only: a resemblance which cannot be given a formal structure which reproduces the successful Schrödinger representation. (It seems that Dirac was aware of the possible problems of this resemblance because he writes: "... the quantum-brackets or at any rate the simpler ones of them, have the same values as the corresponding classical Poisson brackets", [Dirac, p.87].) This puts us back at the beginning when we asked: how does classical mechanics guide us? Canonical quantization is a successful prescription, but it lacks a more rigorous framework which gives us insight in why it works.

In 1970 Kostant and Souriau again considered the question how to pass from classical mechanics to quantum mechanics and they introduced (independently) a theory which is nowadays known as geometric quantization [Kostant 1970; Souriau 1970]. With this theory they (and nowadays many others) try to give a prescription how to pass from classical mechanics to quantum mechanics and they try to formulate the extra ingredients necessary for this passage. The aim of these notes is to give an introduction to this theory with the accent on (heuristic) arguments why it is reasonable to do it the way one does. However, it is not the intention to give a full account of the theory as it is developed till now and especially the so-called Blattner-Kostant-Sternberg-Kernel will not be discussed; the reader is referred to the table of contents to see what topics are treated.

The starting point of geometric quantization is the description of classical mechanics in the Hamilton formalism, i.e. one starts with the phase-space M of the system together with a symplectic form ω on M and a (not yet specified) set $\mathcal{C}$ of classical observables (= functions on M) which is not necessarily closed under the Poisson brackets associated with ω. With these ingredients one tries to find a Hilbert space $\mathcal{H}$ and a map δ such that $\mathcal{H}$ and $\delta(\mathcal{C})$ give the quantum description of this system.

The first step in this construction is the so-called prequantization (discovered by Van Hove [Van Hove 1951] and independently rediscovered by Kostant and Souriau) where a $\hat{\mathcal{H}}$ and $\hat{\delta}$ are constructed which satisfy conditions (Qi)-(Qiv) of our quantization scheme but which fail condition (Qv); in principle prequantization is a slight modification of the notion of Hamilton vector fields on M.

The second step is the introduction of a polarization in order to reduce the size of $\hat{\mathcal{H}}$: heuristically $\hat{\mathcal{H}}$ consists of functions on the phase space M and a polarization reduces this to functions on a configuration space; the introduction of a polarization also gives us a way to define the set $\mathcal{C}$ of directly quantizable observables. Unfortunately in many cases $\mathcal{C}$ does not contain the Hamilton function: one has to use the Blattner-Kostant-Sternberg-Kernel to derive the quantum hamiltonian.

2 PREQUANTIZATION

First a preliminary remark: throughout the sequel we will use three "examples" of symplectic manifolds. The first one is the simplest case: $M = \mathbb{R}^{2n} = T^*\mathbb{R}^n$ with coordinates $(q^i, p_i)_{i=1}^n$ and $\omega = d(p_i dq^i) = dp_i \wedge dq^i$; the second one is the generalization to the cotangent bundle of an arbitrary configuration space Q: $M = T^*Q$ with local coordinates (q^i) on Q and (q^i, p_i) on M with $\omega = d\theta$, θ the canonical 1-form on M (in local coordinates (q^i, p_i): $\theta = p_i dq^i$); the third one is an arbitrary symplectic manifold (M, ω). To simplify notations we will say the symplectic manifold $M = \mathbb{R}^{2n}$ (resp. $M = T^*Q$) and it should be tacitly understood that the above mentioned symplectic forms are used.

To start the description of prequantization, we note that every symplectic manifold carries a natural volume element ε_ω defined by the symplectic form:

$$\varepsilon_\omega = \frac{1}{n!}(-1)^{\frac{1}{2}n(n-1)}\omega^n, \qquad \dim M = 2n$$

(in local canonical coordinates (q^i, p_i) we have $\omega = dp_i \wedge dq^i$ and $\varepsilon_\omega = dp_1 \wedge \ldots \wedge dp_n \wedge dq^1 n \ldots dq^n$). Hence there is associated to M a Hilbert space $\mathcal{H} = L^2(M)$ of all complex valued square integrable functions on M with the inner product

$$< \phi, \psi > = \int_M \bar{\phi}\psi\varepsilon_\omega$$

Secondly, to every function $f \in C^\infty_{\mathbb{R}}(M)$ (a real valued, infinitely differentiable function on M) there is associated a vector field X_f, called the hamiltonian vector field of f, defined by

$$i_{X_f}\omega + df = 0;$$

in local canonical coordinates (q^i, p_i) X_f is represented by

$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i} .$$

Furthermore (Liouville's theorem) the flow ρ_t associated to the vector field X_f preserves the volume element ε_ω, so to X_f is associated a symmetric operator $\bar{\delta}(f)$ on H defined by:

$$\bar{\delta}(f)\phi = -i\hbar X_f\phi$$

where ϕ lies in a suitable dense subset of H. It can be proved that if X_f is a complete vector field (i.e. ρ_t is a 1-parameter group of diffeomorphisms of M) then $\bar{\delta}(f)$ is an (essentially) self-adjoint operator on H; a sketch of the proof is as follows: if X_f is complete then ρ_t is a 1-parameter group of diffeomorphisms which preserve ε_ω hence ρ_t induces a 1-parameter group of unitary transformations of H and we then apply the theorem of M.H. Stone which gives us the desired result.

Using the formula $[X_f,X_g] = X_{[f,g]}$ (commutator of vector fields on the left, Poisson brackets on the right) and the linearity of X_f in f (over $\mathbb{R}$) we see that the map $\bar{\delta}: f \longmapsto \bar{\delta}(f)$ restricted to functions with complete hamilton vector fields satisfies the quantization conditions (Qi)-(Qiii), but it fails condition (Qiv): the constant function 1 has a zero hamilton vector field hence $\bar{\delta}(1) = 0$.

Following [Woodhouse] we try to correct this failure by redefining $\bar{\delta}$:

$$\bar{\delta}(f)\phi = -i\hbar X_f\phi + f\phi$$

but then condition (Qiii) is no longer true. Even this problem can be solved; for the sake of simplicity we assume for the moment that $M = T^*Q$, $\omega = d\theta$, then a correct definition is given by:

$$\bar{\delta}(f)\phi = -i\hbar X_f\phi - \theta(X_f)\phi + f\phi .$$

This definition now satisfies conditions (Qi)-(Qiv) and again one can prove that if X_f is complete then $\bar{\delta}(f)$ is (essentially) self-adjoint (see the end of this section).

In the general case the symplectic form ω need not be exact and we can find only locally a 1-form θ such that $\omega = d\theta$; hence we have to investigate the influence of a different choice of symplectic potential θ. Suppose $\omega = d\theta = d\hat{\theta}$ then there exists (locally) a real function u: $\hat{\theta} = \theta + du$ (Poincaré's lemma) and we have:

$$\bar{\delta}(f)_\theta \phi = -i\hbar X_f \phi - \theta(X_f)\phi + f\phi$$

and

$$\begin{aligned}\bar{\delta}(f)_{\hat{\theta}} \phi &= -i\hbar X_f \phi - \hat{\theta}(X_f)\phi + f\phi \\ &= \exp(iu/\hbar)\{-i\hbar X_f(\exp(-iu/\hbar)\phi) \\ &+ (f-\theta(X_f))\exp(-iu/\hbar)\phi\} \\ \Longleftrightarrow \exp(-iu/\hbar)\bar{\delta}(f)_{\hat{\theta}}\phi &= \bar{\delta}(f)_\theta\{\exp(-iu/\hbar)\phi\}\end{aligned}$$

From the last equality we see that if we introduce a gauge transformation $\phi \longmapsto \exp(iu/\hbar)\phi$ associated to a change in symplectic potential $\theta \longmapsto \theta + du$, then the definition of $\bar{\delta}_\theta(f)$ and $\bar{\delta}(f)_{\hat{\theta}}$ coincide on the common (local) domain of θ and $\hat{\theta}$. This implies that ϕ no longer is a function on M, but an object which assigns to each $m \in M$ a complex number, a complex number depending on the choice of a local symplectic potential, in other words: we need a complex line-bundle of which ϕ is a section.

Let us formalize this idea. Suppose we have an open cover $\mathcal{U} = \{U_i \mid i \in I\}$ of an arbitrary symplectic manifold (M,ω) together with a collection $\{\theta_i, U_{ij} \mid i,j \in I\}$ such that θ_i is a symplectic potential defined on U_i $(d\theta_i = \omega)$ and such that

$$\theta_j = \theta_i + du_{ji} \quad \text{on} \quad U_i \cap U_j.$$

We want to know if we can define a line-bundle with gauge transformations $\exp(iu_{ji}/\hbar)$, in other words if we consider local functions ϕ_i on U_i and if we impose $\phi_j = \exp(iu_{ji}/\hbar)\phi_i$ on $U_i \cap U_j$, do we get a consistent definition of an intrinsic object $\phi = (\phi_i)_{i\in I}$?

On a triple intersection $U_i \cap U_j \cap U_k$ we have:

$$\theta_j = \theta_i + du_{ji}, \quad \theta_k = \theta_j + du_{ij}, \quad \theta_i = \theta_k + du_{ik}$$

hence

$$d(u_{kj}+u_{ji}+u_{ik}) = 0 \iff u_{kj} + u_{ji} + u_{ik} \text{ is locally constant.}$$

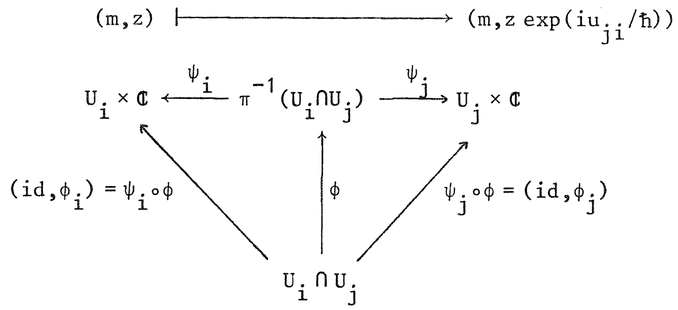
Furthermore:

$$\phi_j = \exp(iu_{ji}/\hbar)\phi_i, \ \phi_k = \exp(iu_{kj}/\hbar)\phi_j, \ \phi_i = \exp(iu_{ik}/\hbar)\phi_k$$

hence for a consistent definition it is necessary that $\exp(i(u_{kj}+u_{ji}+u_{ik})/\hbar) = 1$ on $U_i \cap U_j \cap U_k$ in other words: in order to get a consistent definition of gauge transformations we need the existence of $n_{ijk} \in \mathbb{Z}$:

$$u_{kj} + u_{ji} + u_{ik} = 2\pi n_{ijk} \hbar .$$

On the other hand, this condition is sufficient to guarantee the existence of a complex line-bundle $\pi: L \to M$ over M which has the $\exp(iu_{ji}/\hbar)$ as gauge transformations associated to the cover $\mathcal{U}$ of M, i.e. if $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ is a local trivialization of L then $\psi_j \circ \psi_i^{-1}(m,z) = (m, z\exp(iu_{ji}/\hbar))$. A section ϕ of L (i.e. $\phi: M \to L$ and $\pi \circ \phi = id(M)$) can be identified with a set of functions $\phi_i: U_i \to \mathbb{C}$ such that $\phi_j = \exp(iu_{ji}/\hbar)\phi_i$ when we put $\psi_i \circ \phi(m) = (m, \phi_i(m))$; in a diagram



In this presentation we define the operator $\hat{\delta}(f)$ on the set $\Gamma(L)$ of sections $\phi\colon M \to L$ as follows:

$$\psi = \hat{\delta}(f)\phi \quad \text{is defined by}$$

$$\psi_j = -i\hbar X_f \phi_j - \theta_j(X_f)\phi_j + f\phi_j$$

and, as we have seen, this defines a correct section ψ of L because:

$\theta_k = \theta_j + du_{kj} \Rightarrow \phi_j = \phi_k \exp(-iu_{kj}/\hbar)$

$$\Rightarrow \psi_j = -i\hbar X_f(\phi_k \exp(-iu_{kj}/\hbar)) + (f - \theta_j(X_f))\phi_k \exp(-iu_{kj}/\hbar)$$

$$\Longleftrightarrow \psi_j = \exp(-iu_{kj}/\hbar)\psi_k .$$

So far, so good, we have generalized the notion of a function on M to a section of the line-bundle L and we have defined an operator $\hat{\delta}(f)$ on such sections, but .... the existence of L is subject to the condition that we can find a suitable collection U_i, θ_i, u_{ij} such that $u_{ij} + u_{jk} + u_{ki} = 2\pi n_{ijk}\hbar$ for some $n_{ijk} \in \mathbb{Z}$. One can show that this condition is a condition on ω, a condition which is usually stated as: ω should define an integral cohomology class; in geometric quantization the (necessary) assumption is that this integrality condition on ω is satisfied (notice that for exact symplectic forms ω (as is the case for $M = T^*Q$, $\omega = d\theta$) this condition is automatically satisfied, since then we need only one set $U_i = M$ and do not need any gauge transformations).

Strange as this condition may seem, it can be related to the quantization condition on energy levels (see [Guillemin & Sternberg]) and given the existence of L one can give a (different) geometric interpretation of the Feynman path integral in terms of geometric quantization (see [Simms, LNM 836]).

Before we turn our attention to the question of self-adjoint operators on Hilbert spaces, we first investigate some properties of L related to the definition of $\hat{\delta}(f)$. $\hat{\delta}(f)$ is defined on sections $\phi \in \Gamma(L)$ by

$$\psi = \hat{\delta}(f)\phi \Longleftrightarrow \psi_j = -\left(X_f - \frac{i}{\hbar}\theta_j(X_f)\right)\phi_j + f\phi_j$$

and we might define for $\xi \in V(M)$ = the set of all complex vector fields on M, an operator ∇_ξ on $\Gamma(L)$ by

$$\psi = \nabla_\xi\phi \Longleftrightarrow \psi_j = \xi\phi_j - \frac{i}{\hbar}\theta_j(\xi)\phi_j .$$

We then claim that ∇ is a correctly defined connection on L.

<u>PROPOSITION</u>: *for* $\xi, \eta \in V(M)$, $f, g \in C^\infty(M)$, $\phi, \psi \in \Gamma(L)$ *we have*:

(i) $\nabla_{f\xi+g\eta}\phi = f\nabla_\xi\phi + g\nabla_\eta\phi$

(ii) $\nabla_\xi(f\phi+g\psi) = \nabla_\xi f\phi + \nabla_\xi g\psi = (\xi f)\phi + f\nabla_\xi\phi + (\xi g)\psi + g\nabla_\xi\psi$

where the multiplication by functions $f \in C^\infty(M)$ *is always pointwise, e.g.* $(f\phi)(m) = f(m)\phi(m)$.

Furthermore, we can compute the curvature of this connection:

<u>PROPOSITION</u>: $\operatorname{curv}(\nabla)(\xi,\eta)\phi \overset{\text{det}}{=\!=} i(\nabla_\xi\nabla_\eta - \nabla_\eta\nabla_\xi - \nabla_{[\xi,\eta]})\phi = \frac{1}{\hbar}\omega(\xi,\eta)\phi$
in other words: the curvature of ∇ *is* $\omega/\hbar$.

In terms of this connection (which we will use throughout the sequel) the definition of $\hat{\delta}(f)$ becomes:

$$\hat{\delta}(f)\phi = -i\hbar\nabla_{X_f}\phi + f\phi.$$

Let us now turn to the next question: can we construct a Hilbert space $\hat{H}$ out of $\Gamma(L)$ (and are the $\hat{\delta}(f)$ (essentially) self-adjoint operators)? The answer is yes, and the construction is as follows: we define a hermitian structure (= complex inner product) on the fibres of L by:
if $\ell_1, \ell_2 \in \pi^{-1}(m)$ then $\psi_i(\ell_j) = (m,z_j)$, $j = 1,2$ and we put

$$(\ell_1,\ell_2) \overset{\text{def}}{=\!=} \bar{z}_1 z_2 \in \mathbb{C} .$$

This definition is independent of the choice of the local chart U_i because the gauge transformations are unitary:

$$(m,z) \xmapsto{\psi_j\psi_i^{-1}} (m, z\exp(iu_{ji}/\hbar))$$

$$\Rightarrow \bar{z}_1 z_2 \longmapsto (\overline{z_1\exp(iu_{ji}/\hbar)})z_2\exp(iu_{ji}/\hbar) = \bar{z}_1 z_2.$$

With this hermitian structure $(\ ,\ )$ on the fibres of L we can define a map which assigns to each pair $\phi, \psi \in \Gamma(L)$ a function (ϕ,ψ) on M by:

$$(\phi,\psi)(m) = (\phi(m), \psi(m)),$$

and we note that this map satisfies the relation: if ξ is a real vector field on M, $\xi \in V_{\mathbb{R}}(M)$ (i.e. $\xi_m \in T_mM \subset (T_mM)^{\mathbb{C}}$) then

$$\xi(\phi,\psi) = (\nabla_\xi\phi,\psi) + (\phi,\nabla_\xi\psi);$$

for arbitrary complex vector fields $\xi \in V(M)$ we have $\xi(\phi,\psi) = (\nabla_{\bar{\xi}}\phi,\psi) + (\phi,\nabla_\xi\psi)$. This relation, which is a compatibility relation between the connection ∇ and the hermitian structure $(\ ,\ )$, expresses the fact that the inner product in the fibres is invariant under parallel transport by ∇ along a curve in M.

We now define the (pre) Hilbert space $\hat{H}$ by:

$$\hat{H} = \{\phi \in \Gamma(L) \mid \int_M (\phi,\phi)\varepsilon_\omega < \infty\}$$

with the inner product

$$<\phi,\psi> = \int_M (\phi,\psi)\varepsilon_\omega .$$

The "operators" $\hat{\delta}(f)$ act on $\hat{H}$ by

$$\hat{\delta}(f)\phi = -i\hbar\nabla_{X_f}\phi + f\phi$$

and one can prove (as already said) that if X_f is a complete vector field on M then $\hat{\delta}(f)$ is an (essentially) self-adjoint operator on $\hat{H}$ (see the end of this section).

To summarize our results, assuming that the symplectic form ω satisfied some integrality condition, we constructed a complex line-bundle $\pi: L \to M$ together with a connection ∇ and a compatible hermitian structure $(\, , \,)$ such that the curvature of ∇ equals $\omega/\hbar$. What we have "proved" is in fact part of a theorem of A. Weil (see [Weil] and [Kostant]):

<u>THEOREM</u> (A. Weil): *given a symplectic manifold* (M,ω), *there exists a complex line-bundle* $\pi: L \to M$ *with connection and compatible hermitian structure with the extra condition* $curvature(\nabla) = \omega/\hbar$ *if and only if* ω *satisfies the integrality condition. Furthermore, the various essentially different choices of* $(L,\nabla, (\, , \,))$ *are parametrized by* $H^1(M,S^1)$ *(= the first cohomology group of* M *with values in* $S^1 \subset \mathbb{C}$*); in particular, if* M *is simply connected then* $(L,\nabla, (\, , \,))$ *is unique up to equivalence (if it exists).*

Using this line-bundle L we constructed a Hilbert space $\hat{H}$ out of $\Gamma(L)$ and with the aid of the connection we defined a map $\hat{\delta}$; the pair $(\hat{H},\hat{\delta})$ satisfies the quantization conditions (Qi)-(Qiv) (at least if we restrict $\hat{\delta}$ to functions f for which X_f is complete). However, when we prequantize $M = \mathbb{R}^{2n}$, we do not get the Schrödinger quantization, due to the fact that $(\hat{H},\hat{\delta})$ does not satisfy Qv.

<u>EXAMPLE</u>: $M = \mathbb{R}^{2n}$, $\omega = dp_i \wedge dq^i$. Since ω is exact ($\omega = d\theta$, $\theta = p_i dq^i$ globally) the integrality condition is satisfied and we can take for L the trivial line-bundle $L = \mathbb{R}^{2n} \times \mathbb{C}$; a section ϕ can be identified with a complex function f by

$$\phi(m) = (m, f(m)) \in \mathbb{R}^{2n} \times \mathbb{C} .$$

The connection ∇ is given in terms of these functions by

$$\nabla_\xi f = \xi f - \frac{i}{\hbar}\,\theta(\xi) f,$$

and the compatible inner product by $(f_1,f_2) = \bar{f}_1 f_2$, hence $\hat{H}$ is (isomorphic to) $L^2(\mathbb{R}^{2n})$.

Let us investigate how the observables q^i and p_i are represented:

$$\left.\begin{aligned} \hat{\delta}(q^i)f &= i\hbar\frac{\partial f}{\partial p_i} + q^i f \\ \delta(p_i)f &= -i\hbar\frac{\partial f}{\partial q^i} \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \hat{\theta}(q^i) &= i\hbar\frac{\partial}{\partial p_i} + q^i \\ \hat{\delta}(p_i) &= -i\hbar\frac{\partial}{\partial q^i} \end{aligned}$$

From these calculation we see that prequantization does not give the Schrödinger quantization, and in fact $\hat{H}$ is reducible under the action of $\hat{\delta}(q^i)$ and $\hat{\delta}(p_i)$ (see [Van Hove] for more details).

We conclude this section with an alternative description of $\hat{H}$ and $\hat{\delta}$ which is sometimes useful. Consider the bundle $\pi: L^* \to M$ over M where L^* equals L without the zero section (equivalently: L^* is the principle fibrebundle associated with L with structure group $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$, or equivalently: L^* is the $\mathbb{C}^*$ bundle over M with the same gauge transformations as L), then L^* has local charts $U_i \times \mathbb{C}^*$ and transition functions

$$(U_j \cap U_i) \times \mathbb{C}^* \subset U_i \times \mathbb{C}^* \longrightarrow (U_i \cap U_j) \times \mathbb{C}^* \subset U_j \times \mathbb{C}^*$$

$$(m,z) \longmapsto (m, z\exp(iu_{ji}/\hbar)) .$$

On L^* we consider the set K of functions $\tilde{\phi}$ on L^* satisfying the relation: $\ell \in L^*$, $z \in \mathbb{C}^* \Rightarrow \tilde{\phi}(z\ell) = z^{-1}\tilde{\phi}(\ell)$:

$$K = \{\tilde{\phi}: L^* \to \mathbb{C} \mid \tilde{\phi}(z\ell) = z^{-1}\tilde{\phi}(\ell)\},$$

and we define the map $\sim : \Gamma(L) \to K$, $\phi \longmapsto \tilde{\phi}$ by the following proces: if $\ell \in \pi^{-1}(m) \in L^*$ $(\Rightarrow \ell \neq 0)$ then there exists a unique element $\tilde{\phi}(\ell) \in \mathbb{C}$ such that $\phi(m) = \tilde{\phi}(\ell)\ell \in L$; this function $\tilde{\phi}(\ell)$ obviously satisfies $\tilde{\phi}(z\ell) = z^{-1}\tilde{\phi}(\ell)$ because:

$$\tilde{\phi}(z\ell)z\ell = \phi(m) = \tilde{\phi}(\ell)\ell \Rightarrow \tilde{\phi}(z\ell)z = \tilde{\phi}(\ell).$$

Furthermore, the map $\sim$ is obviously injective, but also surjective because of the inverse construction

$$\phi(m) := \tilde{\phi}(\ell)\ell, \qquad m = \pi(\ell).$$

Thus, $\Gamma(L)$ is equivalent to K; in order to give another description of K we introduce three vector fields on L^*: Z, $\bar{Z}$ and E defined on the local charts $U_i \times \mathbb{C}^*$ with coordinates $z = x + iy = re^{i\theta}$ on $\mathbb{C}^*$ $(x,y \in \mathbb{R},\ r \in \mathbb{R}^+,\ \theta \in [0, 2\pi))$ by

$$Z = z\frac{\partial}{\partial z} = (x+iy)\cdot\tfrac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) = \tfrac{1}{2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + \frac{i}{2}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$$

$$\bar{Z} = \bar{z}\frac{\partial}{\partial \bar{z}} = (x-iy)\cdot\tfrac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \tfrac{1}{2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - \frac{i}{2}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$$

$$E = i(Z-\bar{Z}) = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \frac{\partial}{\partial \theta}$$

and we note that these definitions are correct, i.e. they coincide on the intersection of two charts. With these definitions we claim:

$$K = \{\tilde{\phi}: L^* \to \mathbb{C} \mid Z\tilde{\phi} = -\tilde{\phi},\ \bar{Z}\tilde{\phi} = 0\}.$$

Finally, we introduce the connection 1-form α on L^*, i.e. the connection 1-form in the principle fibrebundle L^* associated with the connection ∇ in the vector bundle L. In general the connection form takes values in the Lie algebra of the structure group G, here $G = \mathbb{C}^*$ hence α takes complex values. On $U_j \times \mathbb{C}^*$ α is defined by:

$$\alpha_j = \frac{1}{\hbar}\theta_j + i\frac{dz}{z} = \frac{1}{\hbar}\theta_j + \frac{ydx-xdy}{x^2+y^2} + i\frac{xdx+ydy}{x^2+y^2}$$
$$= \frac{1}{\hbar}\theta_j - d\theta + i\frac{dr}{r}$$

and again this definition is independent of the local chart. With these ingredients we can define the action of ∇_ξ on sections ϕ in terms of K. Let ξ be a real vector field on M then define the real vector field $\tilde{\xi}$ on L^* by

(i) $\pi_*\tilde{\xi} = \xi$

(ii) $\alpha(\tilde{\xi}) = 0$ (i.e. $\tilde{\xi}$ is a horizontal lift of ξ).

These conditions determine $\tilde{\xi}$ completely: on a local chart $U_j \times \mathbb{C}^*$ $\tilde{\xi}$ is given by

$$\tilde{\xi}_j = \xi + \frac{1}{\hbar}\theta_j(\xi)E = \xi + \frac{1}{\hbar}\theta_j(\xi)\frac{\partial}{\partial\theta}.$$

When we extend this definition to complex vector fields by linearity, we get the following proposition:

<u>PROPOSITION</u>: *if* $\xi \in V(M)$, $f \in C^\infty(M)$, $\phi \in \Gamma(L)$ *then*

$$\widetilde{\nabla_\xi\phi} = \tilde{\xi}\tilde{\phi} \quad \textit{and} \quad \widetilde{f\phi} = f\tilde{\phi}$$

where $(f\tilde{\phi})(\ell) = f(\pi(\ell))\tilde{\phi}(\ell)$ *is again the pointwise multiplication.*

With this proposition we can already translate the action of $\hat{\delta}(f)$ $(f \in C^\infty_{\mathbb{R}}(M))$ in terms of K but a more elegant description is possible if we realize that $Z\tilde{\phi} = -\tilde{\phi}$, $\bar{Z}\tilde{\phi} = 0$ and $\widetilde{f\phi} = f\tilde{\phi}$ imply that $\widetilde{f\phi} = (ifE)\tilde{\phi}$. With this observation we define a map $f \longmapsto \eta_f$ from observables to real vector fields on L^* by:

$$\eta_f = \tilde{X}_f - \frac{1}{\hbar}fE \qquad (= X_f + \frac{1}{\hbar}(\theta_j(X_f) - f)\frac{\partial}{\partial\theta} \text{ on } U_j \times \mathbb{C}^*)$$

and we claim that with these definitions the following proposition holds.

PROPOSITION: *the map* $f \longmapsto \eta_f$ *is injective and:*

(i) $\hat{\delta}(f)\phi = -i\,\eta_f\tilde{\phi}$

(ii) $\eta_{[f,g]} = [\eta_f, \eta_g]$

This proposition gives us an elegant translation of $\hat{H}$ and $\hat{\delta}$ in terms of K (N.B. in [Simms & Woodhouse] the set $\{\eta_f \mid f: M \to \mathbb{R}\}$ is described completely in terms of L^*, α and the inner product on $L^* \subset L$).

We now sketch the proof that if X_f is complete then $\hat{\delta}(f)$ is self-adjoint, a proof which we will illustrate with the case $M = T^*Q$. If X_f is complete, then it can be proved that η_f is complete too; in that case the 1-parameter group of diffeomorphisms σ_t associated to η_f generates a unitary group of transformations of K (with the inner product inherited from $\Gamma(L)$) by $\tilde{\phi} \longmapsto \tilde{\phi} \circ \sigma_t$. It then follows from the last proposition and Stone's theorem that $\hat{\delta}(f)$ is the generator of this group and hence self-adjoint.

EXAMPLE: $M = T^*Q$, $\omega = d\theta$. Since ω is exact we may take $L = M \times \mathbb{C}$, $L^* = M \times \mathbb{C}^*$ together with the usual inner product $((m,z_1), (m,z_2)) = \bar{z}_1 z_2$.

Suppose X_f is complete and denote the associated flow by ρ_t then η_f is given by

$$\eta_f = X_f + A_f \frac{\partial}{\partial\theta} , \qquad A_f = \frac{1}{\hbar}(\theta(X_f) - f) .$$

If we denote the flow of η_f by σ_t then the action of σ_t on L^* is given by:

$$\sigma_t(m,z) = (\rho_t(m), z \cdot \exp(i \int_o^t A_f(\rho_s(m))ds))$$

and from this expression it is an elementary calculation to show that σ_t is a 1-parameter group of diffeomorphisms. With the aid of σ_t we can define an action $\hat{\rho}_t$ on elements of $\hat{H}$ ($\hat{H} \cong L^2(M)$ "$\subset$" $\Gamma(L)$) via K by:

$$\widetilde{\hat{\rho}_t\phi} = \tilde{\phi} \circ \sigma_t$$

whence it follows that

$$(\hat{\rho}_t\phi)(m) = \phi(\rho_t(m))\exp(-i \int_o^t A_f(\rho_s(m))ds)$$

an expression which is unequal to $\phi \circ \rho_t$, but it is a map which preserves the inner product:

$$(\hat{\rho}_t\phi, \hat{\rho}_t\psi) = (\phi,\psi) \circ \rho_t$$

because the extra phase factor cancels. Now ε_ω is invariant under ρ_t (Liouville!) hence $\hat{\rho}_t$ is a unitary group on $\hat{H}$, "hence" its generator $\hat{\delta}(f)$ is self-adjoint by Stone's theorem.

<u>Bibliographical note</u>: the introduction of the prequantization line-bundle L as given in this section is rather clumsy; there exist nicer ways of introducing L (which are, however, more complicated than the one given above): (1) L figures in the interpretation of the symplectic cocycle associated

to a dynamical group acting on a symplectic manifold, a cocycle which indicates whether this group action can be lifted to L or not (see [Souriau, Ch.II, §11] and [Simms & Woodhouse, App.C]). This interpretation is closely related to the theorem of Wigner in quantum mechanics on projective representations of Lie groups in Hilbert spaces. (2) The construction of L is closely related to the third theorem of S. Lie on the existence of a global Lie group for any finite dimensional Lie algebra (see [Van Est] and [Almeida & Molino]).

3 REAL POLARIZATIONS

If we compare for a particle in $\mathbb{R}^n$ the prequantization with the Schrödinger quantization, then the Hilbert space of the first consists of functions of the q^i and p_i simultaneously, in the second case the Hilbert space consists of functions depending on the q^i only. The obvious way to derive the Schrödinger quantization from prequantization is to restrict the attention to functions on M which are independent of the coordinates p_i, but .... then they no longer belong to the Hilbert space of prequantization (except if they are identically zero) because the integral over the p_i diverges!

However, if we restrict our attention to functions independent of p_i and integrate over the q^i ($\mathbb{R}^n$) instead of over the q^i and p_i ($\mathbb{R}^{2n} = M$), then we get the Schrödinger quantization.

We can apply the same reasoning to $M = T^*Q$ with coordinates q^i on Q and associated coordinates p_i in the fibres of $T^*Q \to Q$: we restrict our attention to functions on M which are independent of the p_i (i.e. constant on the fibres of $T^*Q \to Q$) and we integrate over Q instead of over M. Here we encounter our first problem: how do we integrate functions over Q: in general we do not have a natural volume form on Q ! But there are more problems when we try to extend the same reasoning to an arbitrary symplectic manifold: for an arbitrary symplectic manifold (M,ω) one could use a (local!) canonical coordinate system (q^i,p_i) (i.e. $\omega = dp_i \wedge dq^i$) and say: we look at functions independent of the p_i, but this depends highly on the choice of the local canonical coordinates; furthermore: one constructs the prequantization Hilbert space not out of functions on M but out of sections of a line-bundle L which is in general not a trivial one (as it can be for $M = T^*Q$).

Even more: if we have solved these problems, there remains the problem: how do we "integrate over Q"? We see that, in order to generalize "the obvious way" from prequantization to Schrödinger quantization for $M = T^*\mathbb{R}^n$, we have to answer four questions:

1. How do we define "fibres" in a symplectic manifold (M,ω) ?
2. How do we define the equivalent of a configuration space Q ?
3. How do we define sections of L "constant along the fibres"?
4. How do we integrate over this "configuration space"?

In this section we will give an answer to the first three questions, the answer to the last question is the subject of section 5.

We (have to) start with a few preliminary notations and definitions: a (C^∞) distribution D on a manifold M is a "map" which assigns to each point $m \in M$ a linear subspace D_m of T_mM such that:

(i) $k = \dim D_m$ = constant (independent of $m \in M$)

(ii) $\forall m_o \in M \; \exists U$ neighbourhood of m_o and vector fields $X_1, \ldots, X_k$ defined and independent on U such that $\{X_1|_m, \ldots, X_k|_m\}$ span D_m for each $m \in U$.

If X is a vector field defined on an open set $O \subset M$ then we use the abbreviation $X \in D$ for the formula $\forall m \in O : X_m \in D_m$.

A distribution is called integrable if for each point $m_o \in M$ there exists a submanifold N of M such that: $m_o \in N$, $\dim N = k$ and $\forall m \in N : T_mN = D_m$. A necessary and sufficient condition for a distribution to be integrable is given by Frobenius' theorem: D is integrable if and only if $\forall X,Y$ vector fields on M: $X,Y \in D \Rightarrow [X,Y] \in D$.

An integrable distribution is also called a foliation; the maximal connected integral submanifolds (i.e. submanifolds N of M such that $T_mN = D_m$) are called the leaves of the foliation. It sometimes happens that the set of all leaves of a foliation D on a manifold M (denoted by M/D) can be given the structure of a manifold in such a way that the natural projection $\pi: M \to M/D$ is a (C^∞) submersion; in such a case the foliation is called reducible (N.B. this is an unusual definition in foliation theory!).

With these notations and definitions we can define a special distribution D^v (where v stands for vertical) on $M = T^*Q$ (with coordinates q^i on Q and associated coordinates p_i in the fibres of $T^*Q \to Q$): D^v_m is spanned by the vectors $\left\{\frac{\partial}{\partial p_1}\Big|_m, \ldots, \frac{\partial}{\partial p_n}\Big|_m\right\}$. It is easy to show that this distribution is integrable: the leaves of D^v are just the fibres of $\pi: T^*Q \to Q$, i.e. the maximal connected integral manifolds are defined by $\{q^i = \mathrm{const}^i,\ i = 1,\ldots,n\}$; furthermore, the foliation D^v is reducible: $M/D^v = Q$.

We see that in the case $M = T^*Q$ we can define fibres and Q by means of a reducible foliation on M; in the general case we want to do the same, but ... what restrictions do we have to impose on a reducible foliation D on M in order to imitate as close as possible the case $M = T^*Q$? The condition $\dim D = n$ (where $\dim M = 2n$) is obvious; if we wish that there exist local canonical coordinates (q^i, p_i) such that D is spanned (locally) by $\left\{\frac{\partial}{\partial p_i}\right\}$ then it is necessary that D_m is maximal isotropic with respect to ω_m, a fact which needs some explanation. A linear subspace D_m of T_mM is called isotropic with respect to ω_m if $X,Y \in D_m \Rightarrow \omega(X,Y) = 0$; one can show that if D_m is isotropic then $\dim D_m \leqslant n$; if $\dim D_n = n$ then D_m is called "maximal isotropic" or "lagrangian". Now it is easy to show that the subspace D_m of T_mM spanned by $\left\{\frac{\partial}{\partial p_i}\right\}$ (where (q^i, p_i) are canonical coordinates: $\omega = dp_i \wedge dq^i$) is an isotropic subspace hence lagrangian (≡ maximal isotropic) because $\dim D_m = n$. Thus if D is spanned locally by $\left\{\frac{\partial}{\partial p_i}\right\}$ for canonical coordinates (q^i, p_i) then D_m is lagrangian.

On the other hand one can prove the converse: if D is a lagrangian foliation of M then there exist everywhere local canonical coordinates

(q^i, p_i) such that D_m is spanned by $\left\{\frac{\partial}{\partial p_i}\right\}$ or equivalently the leaves are defined (locally) by q^i = const. (see for instance [Woodhouse §4.4]). These facts lead us to the defintion of "fibres" in a general symplectic manifold (M,ω) as a so-called real polarization:

DEFINITION: a real polarization D is a foliation D on M such that

$$\left.\begin{array}{ll}\text{(i)} & \dim D = n \\ \text{(ii)} & D \text{ is isotropic}\end{array}\right\} \Longleftrightarrow D \text{ is lagrangian.}$$

A polarization D is called reducible if the underlying foliation is reducible; in that case M/D is called the generalized configuration space.

EXAMPLE 1: $M = T^*\mathbb{R}^n$, $D = D^h$ (the so-called horizontal polarization): D^h_m is spanned by $\left\{\frac{\partial}{\partial q^i}\Big|_m , i = 1,\dots,n\right\}$. D^h is a reducible polarization: the leaves are defined globally by the equations p_i = const; $M/D^h \cong \mathbb{R}^n$. Functions constant on the fibres are in this case functions of the p_i only, which will lead us to the momentum representation of the Schrödinger quantization.

EXAMPLE 2: $M = T^*Q$, $D = D^v$ (the so-called vertical polarization): D^v_m is spanned by $\left\{\frac{\partial}{\partial p_i}\right\}$. Here again D^v is a reducible polarization and $M/D^v \cong Q$.

EXAMPLE 3: $M = \mathbb{R}^2 \setminus \{(0,0)\}$ with coordinates $(q,p) \neq (0,0)$, $\omega = dp \wedge dq$ and we define a polarization D by D_m is spanned by the vector $\left(q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}\right)$. Because D is 1-dimensional it is automatically integrable (see Frobenius) and lagrangian $(\dim M = 2)$. The leaves of D are circles around $(0,0)$ and we have $M/D \cong \mathbb{R}^+ = \{ r \in \mathbb{R} \mid r > 0\}$, so D is a reducible (real) polarization. However, $T^*(M/D) \cong \mathbb{R}^+ \times \mathbb{R}$ which is not diffeomorphic to M (M has a hole!).

Example 3 shows that in general it is not true that if D is a reducible polarization on M then $T^*(M/D) \cong M$, which shows that the above mentioned proposition about local representations of D cannot be extended to global representations unless more information is available.

We have defined a real polarization as a smooth distribution which is integrable and lagrangian; in the sequel it will be useful to state these conditions on the distribution in terms of (local) functions on M with their Poisson brackets, as is expressed in the following proposition.

PROPOSITION: D *is a real polarization if and only if* D *is a smooth distribution on* M *such that for each* $m_o \in M$ *there is a neighbourhood* U *of* m_o *and* n *independent functions* $f^1, \ldots, f^n$ *on* U (*i.e.* $\forall m \in U$: $df^1, \ldots, df^n$ *are linear independent in* $T^*_m M$) *such that*:

(i) $\forall m \in U : D_m$ *is spanned by* $\{X_{f^1}, \ldots, X_{f^n}\}$ (*the hamiltonian vector fields*)

(ii) *on* U : $[f^i, f^j] = 0$, $i,j = 1, \ldots, n$ (*the Poisson brackets*!).

PROOF: ⟸ $[f^i,f^j] = \omega(X_{f^i},X_{f^j}) = 0 \Rightarrow D_m$ is isotropic; the X_{f^i} are independent $\Rightarrow \dim D_m = n$; on U the integral manifolds of D are defined by $f^i = \text{const.}$ because $df^i(X_{f^j}) = \omega(X_{f^j},X_{f^i}) = [f^j,f^i] = 0$, hence D is integrable.

⟹ by Frobenius there exist n independent local real functions $f^1, \ldots, f^n$ such that the leaves of D are defined locally by $f^i = \text{const.}$ Now $X \in D$ (a vector field!) $\Rightarrow Xf^i = 0 \Rightarrow \omega(X,X_{f^i}) = df^i X = Xf^i = 0 \Rightarrow$ (D is *maximal* isotropic) $X_{f^i} \in D$. Hence by the independence of the X_{f^i} it follows that the $\{X_{f^1},\ldots,X_{f^n}\}$ span D (locally); furthermore, by isotropy of D it follows that $[f^i,f^j] = \omega(X_{f^i},X_{f^j}) = 0$. QED

The preceding discussions show that the first and second question are answered if we have a real reducible polarization D on M: the leaves of D are the "fibres" of M and M/D is the generalized configuration space. Let us now investigate the third question: how to define sections "constant along the fibres". A necessary condition is that the inner product (ϕ_1,ϕ_2) of two sections of L (which is a function on M!) can be viewed as a function on M/D, i.e. (ϕ_1,ϕ_2) should be constant on the leaves of D or equivalently: $\forall x \in D : X(\phi_1,\phi_2) = 0$. Now remember that we have a connection on L with a compatible inner product, hence:

$$0 = X(\phi_1,\phi_2) = (\nabla_X\phi_1,\phi_2) + (\phi_1,\nabla_X\phi_2).$$

This formula suggests that "sections constant along the fibres" should be translated as:

$$\forall x \in D : \nabla_X\phi = 0,$$

but does this formula coincide with our intuitive notion "ϕ independent of coordinates p_i"? If $M = T^*Q$, $D = D^V$, $L = M \times \mathbb{C}$ we can identify sections ϕ with functions f by $\phi(m) = (m, f(m))$ and then the connection is represented by:

$$\nabla_X f = Xf - \frac{i}{\hbar}\theta(X)f$$

where $\theta = p_i dq^i$ is the canonical 1-form on M. If $X \in D^V$ then $X = a_i^j(q,p)\frac{\partial}{\partial p_i} \Rightarrow \theta(X) = 0$ hence the "section" f is independent of the coordinates p_i iff

$$\forall x \in D^V : Xf = 0$$

$$\Longleftrightarrow \forall x \in D^V : \nabla_X f = 0,$$

so for $M = T^*Q$ the above definition of "sections constant along the fibres" is the correct translation of "functions independent of coordinates p_i".

In the general case, as we will see, it also coincides with our intuition: according to the memorated proposition there exist local canonical coordinates (q,p) such that D is spanned locally by $\left\{\frac{\partial}{\partial p_i}\right\}$. In these coordinates $\omega = d\theta$, $\theta = p_i dq^i$, hence, as can be seen from the construction of L, we can represent sections of L locally by functions f and at the same time the connection ∇ is represented locally by

$$\nabla_X f = Xf - \frac{i}{\hbar}\theta(X)f.$$

We now have exactly the same situation as with $M = T^*Q$ (only now it is locally defined) so "sections constant along the fibres" in the sense " $\forall x \in D : \nabla_X \phi = 0$ " is equivalent to the idea "functions independent of p_i" as desired.

We can summarize our answers as follows: in order to obtain the (equivalent of the) Schrödinger quantization out of prequantization, we need a reducible real polarization D on M, i.e. a smooth distribution D on M such that

$$\forall m_o \in M \ \exists U; \ f^1 \dots f^n : U \to \mathbb{R} :$$

$$\text{(i)} \ \{X_{f^i}\} \text{ span } D \text{ on } U$$

$$\text{(ii)} \ [f^i, f^j] = 0 \text{ on } U.$$

The Hilbert space of the quantum theory then should be constructed from those sections ϕ of L (L = the prequantization line-bundle over M) which satisfy the relation

$$\forall x \in D : \nabla_X \phi = 0.$$

This condition is usually stated in words as: sections of L which are covariantly constant along D (and not as "constant along the fibres" because in general M is not a fibrebundle over M/D). The inner product (ϕ_1, ϕ_2) of two sections of L which are covariantly constant along D then can be interpreted as a function on the generalized configuration space M/D; how we do integrate over M/D will be discussed in section 5.

4 DENSITIES AND ½-DENSITIES: A technical intermezzo

In this section we will study some properties of n-forms on a manifold X (dim X = n) and we will generalize the concept of an n-form on an oriented manifold to the concept of a density on an arbitrary manifold in such a way that both can be integrated over the manifold in question and such that both concepts coincide on oriented manifolds with regard to integration. With the concept of densities we will generalize to manifolds the notion of L^1-functions on $\mathbb{R}^n$ (with Lebesgue measure $\lambda^{(n)}$); with the same technique we also will generalize the notion of L^2-functions on $(\mathbb{R}^n, \lambda^{(n)})$.

Contrary to $\mathbb{R}^n$, where we have the Lebesgue measure, on an arbitrary manifold X no preferred measure (or volume element) is available, hence there is no natural way to define the integral of functions on X over X. However, if the manifold X is oriented, we can define in a consistent way the integral over X of n-forms ω (which we view as sections of the bundle $\Lambda^n(T^*X)$) and, given such an n-form ω we can define the integral of functions f by:

$$I_\omega(f) = \int_M f\omega$$

since if ω is an n-form, $f\cdot\omega$ is one too. It is evident that this integral $I_\omega(f)$ depends on ω and (again) in general no natural choice for ω is available; however, there is a more important restriction: the construction of $I_\omega(f)$ is possible only on oriented manifolds, which implies that X should be orientable! Let us investigate the origin of this restriction and try to avoid it.

Suppose X is a manifold, dim X = n and suppose we have a cover $\mathcal{U} = \{U_i \mid i \in I\}$ of local charts, i.e. there exist $\phi_i\colon U_i \to O_i \subset \mathbb{R}^n$ homeomorphisms of U_i to open sets of $\mathbb{R}^n$ such that $\phi_{ji} := \phi_j \circ \phi_i^{-1}$ (whenever defined) is a C^∞-map. In the sequel we will identify U_i with $O_i \subset \mathbb{R}^n$ and use the standard coordinates x^i $(i = 1,\dots,n)$ of $\mathbb{R}^n$ as (local) coordinates on U_i ; we will denote by $J_{ji}(x)$ the Jacobian matrix of $\phi_{ji}(x)$:

$$(J_{ji})_{pq} = \frac{\partial \phi^p_{ji}}{\partial x^q}.$$

We now introduce the bundle $\pi: F^kX \to X$ of k-frames over X, i.e. the fibre $\pi^{-1}(x)$, denoted by F^k_xX, consists of all sets of k vectors $\xi_1, \dots, \xi_k \in T_xX$ which are linear independent (over $\mathbb{R}$); we define F^kX for all $1 \leqslant k \leqslant n$ for future use, in this section we only need F^nX. If we forget for the moment that the k vectors $\xi_1 - \xi_k$ should be independent, then the bundle F^kX has local charts $U_i \times \mathbb{R}^{nk}$ as follows: in U_i the vectors $\left.\frac{\partial}{\partial x^p}\right|_x$ $(p = 1,\dots,n)$ constitute as basis of T_xX hence each vector ξ_j can be expressed in terms of this basis by

$$\xi_\ell = \xi_{p\ell} \left.\frac{\partial}{\partial x^p}\right|_x \in T_xX \qquad \ell = 1, \dots, k$$

so the point $(x, \xi_1 - \xi_k)$ has coordinates $(x^1, \dots, x^n, \xi_{p\ell}, \ell = 1-k, p = 1-n) \in U_i \times \mathbb{R}^{nk} \subset \mathbb{R}^{n+nk}$; the transition functions between two coordinate charts are given by:

$$\psi_{ji}: U_i \times \mathbb{R}^{nk} \longrightarrow U_j \times \mathbb{R}^{nk}$$

$$(x, \xi_\ell) \longmapsto (\phi_{ji}(x), J_{ji}\xi_\ell)$$

$$\xi_\ell = \xi_{p\ell} \left.\frac{\partial}{\partial x^p}\right|_x \longmapsto J_{ji}\xi_\ell = (J_{ji})_{pq}\xi_{q\ell}\frac{\partial}{\partial x^p} = \frac{\partial \phi^p_{ji}}{\partial x^q} \cdot \xi_{q\ell}\frac{\partial}{\partial x^p}$$

When we remember that the $\xi_1, \dots, \xi_k$ should be independent, then this implies that the local charts are not $U_i \times \mathbb{R}^{nk}$, but $U_i \times V$ where V is an open set in $\mathbb{R}^{nk}$; because the Jacobian J_{ji} has a nowhere zero determinant, a set of k independent vectors is mapped on a set of k independent vectors, hence the above defined transition functions remain valid.

Before we turn our attention to F^nX, it should be noted that F^1X is a well-known bundle: F^1X is nothing more or less than the tangent bundle TX without the zero section, i.e. $F^1_xX = T_xX \setminus \{0\}$.

Now we study F^nX in more detail: the fibres of F^nX are isomorphic to $GL(n,\mathbb{R})$ as can be seen when we note that n independent vectors $\xi_1 - \xi_n \in T_xX$ form a nonsingular $n \times n$ matrix $\xi = (\xi_{p\ell})$ defined by

$$\xi_\ell = \xi_{p\ell} \frac{\partial}{\partial x^p} , \qquad \ell = 1, \ldots, n$$

on a local chart U_i (and vice-versa a nonsingular matrix $(a_{p\ell})$ defines an n-frame $\xi_\ell = a_{p\ell} \frac{\partial}{\partial x^p}$). Furthermore, we can define a right action of $GL(n,\mathbb{R})$ on F^nX by:

$$a \in GL(n,\mathbb{R}) \wedge (x, \xi_1, \ldots, \xi_n) \in F^n_xX$$

$$\Rightarrow \begin{cases} (x,\xi) \cdot a = (x,\xi a) \\ \xi = (\xi_{p\ell}),\ a = (a_{pq}) \Rightarrow \xi a = (\xi_{pm}\, a_{m\ell}) \end{cases}$$

We leave it to the interested reader to prove that this definition of a right-action of $GL(n,\mathbb{R})$ on F^nX is independent of the local chart $U_i \times V$ (which is essentially due to the fact that the Jacobian matrix acts to the left of ξ and this action to the right). With these remarks we now have "proved" that F^nX is the principal $GL(n,\mathbb{R})$ bundle over X associated with TX; the latter because the transition functions (gauge-transformations) are determined by the Jacobian matrix of ϕ_{ji}, or vice-versa TX is the vector bundle associated to the principal bundle F^nX by means of the identity representation of $GL(n,\mathbb{R})$.

Next we turn back to n-forms: an n-form ω is an object which assigns to a point x and a set of n independent vectors $\xi_1 - \xi_n \in T_x X$ a real number $\omega_x(\xi_1,\ldots,\xi_n)$ such that if $\xi_i = a_{ji}\eta_j$ then

$$\omega_x(\xi_1,\ldots,\xi_n) = \det a \cdot \omega_x(\eta_1,\ldots,\eta_n),$$

in other words: ω is a function on $F^n X$ such that

$$a \in GL(n,\mathbb{R}) \wedge (x,\xi) \in F^n X$$
$$\Rightarrow \omega(x,\xi a) = \det a \cdot \omega(x,\xi).$$

The integral $\int_X \omega$ is defined by the following proces: let $\{\rho_i\}$ be a partition of unity subordinated to the coordinate cover $\mathcal{U} = \{U_i \mid i \in I\}$, then $\rho_i \cdot \omega$ is an n-form on $U_i \subset \mathbb{R}^n$ and one defines

$$\int_{U_i} \rho_i \omega = \int_{U_i} \rho_i \omega\left(\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n}\right) d\lambda^{(n)}.$$

The absolute value of this integral is independent of the chosen local coordinates on U_i because of the transformation properties of Lebesgue integrals; it changes sign whenever the determinant of the Jacobian of the coordinate change has an (everywhere) minus sign (it is nowhere zero!). Here we see the origin of our troubles: if we want to define the integral $\int_X \omega$ independent of all choices, it is necessary to have an orientation and to restrict the coordinate changes to those having a positive Jacobian determinant; if so one defines

$$\int_X \omega = \sum_i \int_{U_i} \rho_i \omega = \sum_i \int_{U_i} \rho_i \omega\left(\frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^n}\right) d\lambda^{(n)}$$

reversing the orientation then changes the sign of $\int_X \omega$!

When we have seen this construction and its problems, we define a density ϕ as follows: a density ϕ is a function $\phi\colon F^nX \to \mathbb{R}$ such that:

$$a \in GL(n;\mathbb{R}) \wedge (x,\xi) \in F^nX$$

$$\Rightarrow \phi(x,\xi a) = \phi(x,\xi) \cdot |\det a| .$$

We define the integral of a density ϕ by

$$\int_X \phi = \sum_i \int_{U_i} \rho_i \phi\left(x, \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right) d\lambda^{(n)}$$

where $\{\rho_i\}$ is the afore mentioned partition of unity subordinated to the coordinate cover $\mathcal{U}$. This definition of $\int_X \phi$ is independent of all choices just because of the absolute value in the definition of ϕ.

On oriented manifolds there exists a bijection between n-forms ω and densities ϕ which is given by

$$\phi(x,\xi) = \omega(x,\xi) \cdot \operatorname{sign}(\xi)$$

where $\operatorname{sign}(\xi) = -1$ if ξ has a negative orientation, else $\operatorname{sign}(\xi) = +1$; moreover, in this case we have

$$\int_X \phi = \int_X \omega$$

In conclusion we might say that the concept of a density generalizes the concept of an n-form in such a way that the two concepts coincide on oriented manifolds (with regard to integration).

To give an "explanation" of the name density, we note that if ϕ is a density, then ϕ defines a measure μ on X by

$$\mu(E) = \int_X 1_E \phi$$

where 1_E is the characteristic function of the set E; for this measure we have (for functions f on X):

$$\int_X f\,d\mu = \int_X f\phi \, .$$

To be more specific, consider $X = \mathbb{R}^n$ and ϕ a C^∞-density on $\mathbb{R}^n$ then

$$\int_{\mathbb{R}^n} f\,d\mu = \int_{\mathbb{R}^n} f\tilde{\phi}\,d\lambda^{(n)}$$

where $\tilde{\phi}$ is the function $\tilde{\phi}(x) = \phi\left(x, \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)$.

We now give another description of densities on X, a description analogous to n-forms as sections of the bundle $\Lambda^n(T^*X)$. We define the bundle $\Delta^1 X$ over X as follows: the fibre $\Delta^1_x X$ over $x \in X$ consists of all functions $\phi_x\colon F^n_x X \to \mathbb{R}$ such that $\phi_x(\xi a) = \phi_x(\xi)\,|\det a|$. Because the fibre $F^n_x X$ is isomorphic to $GL(n,\mathbb{R})$, such a function is completely determined by its value on a specific frame $\xi_o \in F^n_x X$. Hence $\Delta^1 X$ is a real line-bundle over X with local charts $U_i \times \mathbb{R}$: (x,t) denotes the functions $\phi_x\colon F^n_x X \to \mathbb{R}$ with

$$\phi_x\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right) = t$$

where $x^1, \ldots, x^n$ are coordinates on U_i; the transition-functions are given by

$$\begin{aligned} \chi_{ji}\colon U_i \times \mathbb{R} &\longrightarrow U_j \times \mathbb{R} \\ (x,t) &\longmapsto (\phi_{ji}(x),\, t \cdot |\det J_{ji}(x)|^{-1}) . \end{aligned}$$

With this definition, a density ϕ on X is just a section of the bundle $\Delta^1 X$:

$$\phi \text{ a density} \iff \phi\colon X \to \Delta^1 X.$$

If $\{\rho_i\}$ is the afore mentioned partition of unity and if ϕ_i is the density on U_i defined by $\phi_i\left(x, \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right) = 1$, then $\phi_o = \Sigma \rho_i \phi_i$ is a well-defined density on X which is nowhere zero: ϕ_o is a global non-vanishing section of $\Delta^1 X$ hence $\Delta^1 X$ is a trivial bundle over X and ϕ_o is a trivializing section. Although we never mentioned it, we always assume tacitly that our manifolds are C^∞-manifolds; because we may assume that $\{\rho_i\}$ is C^∞, the bundle $\Delta^1 X$ is trivial even as C^∞-bundle over X.

With the aid of ϕ_o we can define a bijection between functions f and densities ϕ on X by:

$$\phi = f\phi_o$$

which is a bijection because ϕ_o is nowhere zero; if we denote by μ_o the measure on X defined by ϕ_o then we have:

$$\int_X f \, d\mu_o = \int_X \phi .$$

This formula gives us the opportunity to interpret densities as a generalization of (real) L^1-functions: with this formula we can define "integrable densities": densities for which $\int_X |\phi|$ is finite, compared with integrable functions: functions for which $\int_X |f| d\mu_o$ is finite. However, the class of integrable functions depends on the measure μ_o, contrary to the class of "integrable densities" which is independent of the choice of ϕ_o. It is in this sense that we say: densities on a manifold X generalize functions on $\mathbb{R}^n$ with the Lebesgue measure: out of densities on X we can construct a class of "integrable densities" analogous to the set $L^1_{\mathbb{R}}(\mathbb{R}^n, \lambda^{(n)})$ of (equivalence-classes of real) $\lambda^{(n)}$-integrable functions on $\mathbb{R}^n$.

We now can ask: densities on a manifold X are the generalization of L^1-functions on $\mathbb{R}^n$, does there exist a similar generalization of L^2-

functions? The answer is affirmative and given by the concept of a $\frac{1}{2}$-density. For the sake of generality/simplicity we will define the more general concept of a r-density ($r \in \mathbb{R}$) : a r-density ϕ is a function $\phi: F^nX \to \mathbb{R}$ such that:

$$a \in GL(n,\mathbb{R}) \wedge (x,\xi) \in F^nX$$

$$\Rightarrow \phi(x,\xi a) = \phi(x,\xi) \cdot |\det a|^r$$

(whence a density should be called a 1-density).

Similar to Δ^1X we define the bundle Δ^rX over X as the bundle whose fibres Δ^r_xX consist of all functions $\phi_x : F^n_xX \to \mathbb{R}$ such that

$$\phi_x(\xi a) = \phi_x(\xi) \cdot |\det a|^r .$$

Again this is a line-bundle (because the value of ϕ_x in a particular frame ξ_o determines ϕ_x completely), the sections of Δ^rX are r-densities (and vice-versa) and Δ^rX is a trivial bundle (using the same argument as for Δ^1X).

How does a $\frac{1}{2}$-density generalize L^2-functions? If ϕ_1 and ϕ_2 are $\frac{1}{2}$-densities, then their pointwise product $\phi_1 \cdot \phi_2$ defined by

$$(\phi_1 \cdot \phi_2)(x,\xi) = \phi_1(x,\xi) \cdot \phi_2(x,\xi)$$

is a 1-density which can be integrated over X, hence we have an inner product on the set $\Gamma(\Delta^{\frac{1}{2}}X)$ of all $\frac{1}{2}$-densities on X defined by

$$(\phi_1,\phi_2) = \int_X \phi_1 \cdot \phi_2 .$$

Now if ϕ_o is a trivializing section of $\Delta^{\frac{1}{2}}X$ (i.e. ϕ_o is a nowhere vanishing $\frac{1}{2}$-density) then $\phi_o^2 = \phi_o \cdot \phi_o$ is a (positive) trivializing section of Δ^1X. If we denote by μ_o the measure associated to ϕ_o^2 then for functions $f,g: X \to \mathbb{R}$ we have a bijection between $\frac{1}{2}$-densities and functions:

$$\phi = f\phi_o \ , \qquad \psi = g\phi_o$$

and moreover we have an inner product on the functions given by:

$$(f,g) = \int fg\, d\mu_o = \int \phi\cdot\psi = (\phi,\psi).$$

This formula enables us, analogous to the case of densities, to see the $\frac{1}{2}$-densities on X as a generalization of L^2-functions on $(\mathbb{R}^n,\lambda^{(n)})$.

It should be said that there exists a definition of $\frac{1}{2}$-densities which is independent of the differentiable structure on Q (i.e. without using the bundle F^nX) as can be read in [Abraham & Marsden, §5.4]. Their definition involves the Radon-Nikodym derivative of measures and the class of measures μ which are equivalent (in the sense of absolute continuity) to the Lebesgue measure on every coordinate chart; a $\frac{1}{2}$-density is denoted by the symbol $f\sqrt{d\mu}$. The concept of $\frac{1}{2}$-densities defined in this way (independent of a differentiable structure) is also used in the construction of unitary representations of topological groups.

We finish this section with another generalization: we generalize r-densities to complex tangent vectors and to densities with complex values.

If we use in the definition of F^kX the complexified tangent space $T_xX^{\mathbb{C}}$ instead of T_xX and if we use linear independence over $\mathbb{C}$ instead of over $\mathbb{R}$, then we get the bundle $F^kX^{\mathbb{C}}$ of all complex k-frames over X. Now $F^1X^{\mathbb{C}}$ is the complexified tangent space without the zero section and $F^nX^{\mathbb{C}}$ is the principal $GL(n,\mathbb{C})$ bundle over X associated with $TX^{\mathbb{C}}$, i.e. the fibres $F^n_xX^{\mathbb{C}}$ are isomorphic to $GL(n,\mathbb{C})$ and there is a right-action of $GL(n,\mathbb{C})$ on $F^nX^{\mathbb{C}}$ which gives F^nX the structure of a principal fibrebundle. These facts are a straightforward generalization of the real case F^nX, one only uses the field of complex numbers instead of the (not always explicitly mentioned) reals.

We then define a real r-density as a function $\phi : F^nX^{\mathbb{C}} \to \mathbb{R}$ with the property

$$a \in GL(n,\mathbb{C}) \wedge (x,\xi) \in F^nX^{\mathbb{C}}$$
$$\Rightarrow \phi(x,\xi a) = \phi(x,\xi)\, |\det a|^r .$$

In the same way as before we define the bundle $\Delta^r X$ such that the fibres $\Delta^r_x X$ consist of all functions $\phi_x : F^n_x X^{\mathbb{C}} \to \mathbb{R}$ with the property

$$\phi_x(\xi a) = \phi(\xi)\, |\det a|^r .$$

We then note that $\Delta^r X$ is a real line-bundle over X (because ϕ_x is determined completely by its value on a fixed complex frame) and moreover: it is the same bundle as before, because we may fix its value on a real frame. Hence sections of $\Delta^r X$ represent real r-densities on X viewed as functions on either $F^n X$ or $F^n X^{\mathbb{C}}$ (with certain transformation properties).

The second point we wanted to generalize is the range of the densities: a complex r-density is a function $\phi : F^n X \to \mathbb{C}$ such that

$$a \in GL(n,\mathbb{R}) \wedge (x,\xi) \in F^n X$$
$$\Rightarrow \phi(x,\xi a) = \phi(x,\xi)\, |\det a|^r .$$

We then define the bundle $\Delta^r X^{\mathbb{C}}$ as the bundle whose fibres $\Delta^r_x X^{\mathbb{C}}$ consist of all functions $\phi_x : F^n_x X \to \mathbb{C}$ such that $\phi_x(\xi a) = \phi_x(\xi)\, |\det a|^r$. By now it should be obvious that $\Delta^r X^{\mathbb{C}}$ is a trivial complex line-bundle over X (as $\Delta^r X$ is a trivial real line-bundle over X) and moreover: $\Delta^r X^{\mathbb{C}}$ is the complexification of $\Delta^r X$ (because we can fix the real and complex functions ϕ_x on $F^n_x X$ on the same frame $\xi_o \in F^n_x X$).

In the same way as before we can visualize the complex 1-densities as a generalization of $L^1_{\mathbb{C}}(\mathbb{R}^n, \lambda^{(n)})$ (= the complex valued absolutely integrable functions on $\mathbb{R}^n$). In order to visualize complex valued $\frac{1}{2}$-densities as a generalization of complex L^2-functions on $\mathbb{R}^n$, we need to define the inner product slightly different from before (just adding a complex conjugation!): if ϕ and ψ are complex $\frac{1}{2}$-densities we define

$$(\phi,\psi) = \int_X \bar{\phi}\cdot\psi$$

where we use complex conjugation in the first factor as is customary in physics.

As in the case of real densities, we can define complex densities as functions $\phi\colon F^nX^{\mathbb{C}} \to \mathbb{C}$ such that:

$$a \in GL(n,\mathbb{C}) \wedge (x,\xi) \in F^nX^{\mathbb{C}}$$

$$\Rightarrow \phi(x,\xi a) = \phi(x,\xi)\,|\det a|^r$$

but as in the real case, this does not change the definition of $\Delta^rX^{\mathbb{C}}$, hence sections of $\Delta^rX^{\mathbb{C}}$ represent complex r-densities viewed as complex functions on either F^nX or $F^nX^{\mathbb{C}}$.

Nota Bene: in the sequel we will omit the superscript $\mathbb{C}$ in $\Delta^rX^{\mathbb{C}}$, so Δ^rX will always represent complex r-densities unless stated otherwise explicitly.

5 -½-D-DENSITIES

In section 3 we saw that, in order to mimic the Schrödinger quantization, we needed a reducible real polarization D on our symplectic manifold (M,ω). With this polarization one can define sections of L (the prequantization line-bundle) such that the inner product of two of them is a function on M/D (= the generalized configuration space). The mimicry would be complete if we had a natural way to integrate over M/D, which we do not! This implies that we need extra structure to do it; the easiest extra structure should be a measure or a density on M/D, but (in general) there is no natural choice, so we need something else.

Let s_1 and s_2 be arbitrary sections of L which are covariantly constant along D, then (s_1,s_2) "is" a function on M/D. As we have seen, the introduction of a density at this point involves an arbitrary choice; a more natural way is the following: do not add the additional structure at the level of the inner product (s_1,s_2), but add some structure to the sections s_i themselves such that the inner product (s_1,s_2) turns out as a density on M/D, not a function. The idea to do this, is to multiply a section s_i of L with a ½-density ϕ_i on M/D, then the "product" $(s_1\phi_1,s_2\phi_2) = (s_1,s_2)\bar{\phi}_1\phi_2$ is a density on M/D; the Hilbert space of quantum mechanics then should be constructed out of such products $s\phi$ instead of out the s alone.

Before we try to realize this idea, let us investigate how it looks like in the Schrödinger quantization. In $M = T^*\mathbb{R}^n$, $L = \mathbb{R}^{2n}\times\mathbb{C}$, $D = D^v$ we can identify sections of L with functions on $\mathbb{R}^{2n}$ and sections covariantly constant along D^v are defined globally by $\frac{\partial s}{\partial p_i} = 0$ $(i = 1,\dots,n)$, so in this case (and not in the general case!) sections covariantly constant along D^v can be identified with functions on $M/D = \mathbb{R}^n$ (with coordinates q^i). On $\mathbb{R}^n$ we have a natural way to identify ½-densities with functions by means of the global non-vanishing $\frac{1}{2}$-density ϕ_o defined by

$$\phi_o\left(q^i, \frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}\right) = 1$$

which implies that the measure associated with the 1-density ϕ_o^2 equals the Lebesgue measure on $\mathbb{R}^n$. Hence if s_i is an arbitrary section of L covariantly constant along D^v ($\Longleftrightarrow s_i$ is a function on $\mathbb{R}^n$) and if ϕ_i is an arbitrary $\frac{1}{2}$-density on M/D ($\Longleftrightarrow \phi_i = f_i\phi_o$, f_i a function on $\mathbb{R}^n$) then $(s_1\phi_1, s_2\phi_2) = \overline{s_1 f_1}\, s_2 f_2\, \phi_o^2$ which can be integrated over $\mathbb{R}^n$ and yields:

$$< s_1\phi_1, s_2\phi_2 > = \int_{\mathbb{R}^n} \overline{s_1 f_1}\, s_2 f_2\, d\lambda^{(n)} .$$

If s_i and f_i are arbitrary, then $s_i f_i$ is an arbitrary function on $\mathbb{R}^n$ and we "see" that the associated Hilbert space is (in a not yet defined way) given by L^2 - functions on $\mathbb{R}^n$ as is the case in the Schrödinger quantization. The conclusion is that our idea might lead us to a correct generalization of the Schrödinger quantization.

We now turn back to the realization of the idea in the general case. It seems evident that we need the bundle $L \otimes \Delta^{\frac{1}{2}}(M/D)$ of which sections are written as $s \otimes \phi$ but ... this tensor product IS NOT defined because L and $\Delta^{\frac{1}{2}}(M/D)$ are bundles over different base spaces: L is a line-bundle over M and $\Delta^{\frac{1}{2}}(M/D)$ is a (trivial) line-bundle over M/D. What we have to do is either to interpret sections of L which are covariantly constant along D as sections of some bundle over M/D, or to interpret sections of $\Delta^{\frac{1}{2}}(M/D)$ as sections of a bundle over M. Although the example of $M = T^*\mathbb{R}^*$ suggests that we can interpret sections covariantly constant along D as sections of a bundle over M/D, this is not possible in the general case (as far as I know) and instead one interprets $\frac{1}{2}$-densities on M/D as sections of a (complex trivial line) bundle B^D over M.

Let us give a heuristical outline of the forthcoming program: we establish a bijection between ½-densities on M/D (i.e. section of $\Delta^{\frac{1}{2}}(M/D)$) and a subset of all sections of B^D. We then define (in the next section) the quantum bundle QB over M as $QB = L \otimes B^D$; each section ψ of QB has a (local) representation $\psi = s \otimes \nu$ where ν is a section of B^D and s a section of L. We then define the Hilbert space H as follows: H consists of those sections ψ of QB for which a representation $\psi = s \otimes \nu$ exists with ν the image of a ½-density ϕ on M/D and with s a section of L which is covariantly constant along D *and* for which

$$\int_{M/D} (s,s)\bar{\phi}\phi$$

is finite. The inner product on H is defined by the formula

$$<\psi_1,\psi_2> = \int_{M/D} (s_1,s_2)\bar{\phi}_1\phi_2.$$

So far our program; the rest of this section will be devoted to the definition/construction of the bundle B^D (where D is the symbol of the (real) polarization on (M,ω)) and the correspondence between sections of $\Delta^{\frac{1}{2}}(M/D)$ and sections of B^D.

A ½-density ϕ on M/D is a function which assigns to n (independent) tangent vectors (ξ_i) of $T_x(M/D)$ a complex number $\phi(x,\xi_1,\ldots,\xi_n)$. We are now going to lift the vectors ξ_i to tangent vectors of M and we define a ½-density as a function on this lift. One's first impuls would be to say: such a lift is not unique and indeed if we consider the canonical projection $\pi\colon M \to M/D$ then there do not exist unique vectors $\eta_i \in T_mM$ such that $\pi_*\eta_i = \xi_i \in T_{\pi(m)}(M/D)$. However, there is another way of lifting a frame of $T_{\pi(m)}(M/D)$ to T_mM: a frame of $T_{\pi(m)}(M/D)$ consists of a basis

(ξ_i) of $T_{\pi(m)}(M/D)$ hence the dual basis $(c_i)_{i=1}^n \in T^*_{\pi(m)}(M/D)$ exists: $c_i(\xi_j) = \delta_{ij}$. This basis can be mapped onto n independent vectors of $T^*_m M$ by π^* and then we can transform them back to tangent vectors $\tilde{\xi}_i$ by means of the symplectic form ω: $i_{\tilde{\xi}_j}\omega + \pi^*_m c_j = 0$.

Let us summarize this lift: given n independent vectors $(\xi_i) \in T_{\pi(m)}(M/D)$ we construct n (unique) independent vectors $(\tilde{\xi}_i) \in T_m M$ as follows:

(i) $\quad c_i \in T^*_{\pi(m)}(M/D) \; : \; c_i(\xi_j) = \delta_{ij}$

(ii) $\quad \tilde{\xi}_j \in T_m M \; : \; i_{\tilde{\xi}_j}\omega + \pi^*_m c_j = 0$

Note that we do not have necessarily $\pi_* \tilde{\xi}_i = \xi_i$; in fact we have $\pi_* \tilde{\xi}_i = 0$, i.e. we claim that if (ξ_i) is a basis of $T_{\pi(m)}(M/D)$ then $(\tilde{\xi}_i)$ is a basis of D_m. Before we state this claim as a proposition, we introduce the bundle R over M as the bundle whose fibres R_m consist of all frames of D_m. It follows that R is a subbundle of $F^n M$ (= the bundle of all n-frames on M; $\dim M = 2n$) and moreover: R is a principal $GL(n,\mathbb{R})$ bundle over M (the frames of D_m can be indexed by $GL(n,\mathbb{R})$ just as the frames of $T_{\pi(m)}(M/D)$). Using this terminology, we have constructed a map from $F^n_{\pi(m)}(M/D)$ to R_m: each frame (ξ_i) of $T_{\pi(m)}(M/D)$ maps to a frame $(\tilde{\xi}_i)$ of D_m and now we can state our proposition.

<u>PROPOSITION</u>: *for each* $m \in M$ *the map* $(\xi_i) \longmapsto (\tilde{\xi}_i)$ *is a bijection from* $F^n_{\pi(m)}(M/D)$ *to* R_m.

<u>PROOF</u>: for each $m \in M$ there exist local canonical coordinates (q,p) in which D is spanned by $\left\{\frac{\partial}{\partial p_i}\right\}$; on the neighbourhood U where (q,p) are defined, the leaves of D are defined by $q^i = \text{const}$, hence we may use the coordinates (q) as coordinates on $\pi(U)$, so $\pi(q,p) = (q)$. Using these coordinates, each frame (ξ) of $T_{\pi(m)}(M/D)$ is defined uniquely by

a matrix $g \in GL(n,\mathbb{R})$:

$$\xi_i = \frac{\partial}{\partial q^j}\Big|_{\pi(m)} g_{ji}.$$

If we denote the standard frame $\left(\frac{\partial}{\partial q^j}\Big|_{\pi(m)}\right)$ by (ξ_o) then we have

$$(\xi) = (\xi_o)g$$

where we use the same notation as in section 4. If we denote by (η_o) the "standard" frame of D_m:

$$(\eta_o) = \left(-\frac{\partial}{\partial p_i}\Big|_m\right) = (X_{q^i})$$

then each frame (η) of D_m is characterized by a matrix $h \in GL(n,\mathbb{R})$:

$$\eta_i = -\frac{\partial}{\partial p_j}\Big|_m h_{ji} \iff (\eta) = (\eta_o)h.$$

If we now compute the image of an arbitrary frame $(\xi) = (\xi_o)g$ on $T_{\pi(m)}(M/D)$, then we get:

$$(\xi) = (\xi_o)g \quad \iff \xi_i = \frac{\partial}{\partial q^j} g_{ji}$$

$$\iff c_i = (g^{-1})_{ij} dq^j$$

$$i_{\tilde{\xi}_j} \omega + \pi^* c_j = 0 \quad \Rightarrow \tilde{\xi}_j = -(g^{-1})_{jk} \frac{\partial}{\partial p_k}$$

hence $(\xi) = (\xi_o)g \Rightarrow (\tilde{\xi}) = (\eta_o)h$ with $h = (g^{-1})^T$.
Since the map $g \longmapsto (g^{-1})^T : GL(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R})$ is bijective, the map $(\xi) \longmapsto (\tilde{\xi}) : F^n_{\pi(m)}(M/D) \longrightarrow R_m$ is bijective too. Note that we have proved and passant our claim that $\tilde{\xi}$ is a frame of D_m ! QED

<u>COROLLARY:</u> $g \in GL(n,\mathbb{R}) \Rightarrow (\xi)g = (\tilde{\xi})(g^{-1})^T$.

It is interesting to see how the lift $(\tilde{\xi})$ varies with m while $\pi(m)$ remains constant:

PROPOSITION: *suppose* $(\xi) \in F^n_x(M/D)$, $m_o \in \pi^{-1}(x)$ *and* $\{\eta_i\}$ n *independent locally hamiltonian vector fields defined in a neighbourhood* U *of* m_o *such that* $\{\eta_i\}$ *span* D *on* U *and such that* $(\eta_i|_{m_o}) = (\tilde{\xi})$ (*N.B. this is possible because* D *is a polarisation*!)

Then $(\eta|_{m_o}) = (\tilde{\xi})$ *for all* $m \in U \cap \pi^{-1}(x)$.

PROOF: we may assume that U is small enough so there exist functions e_i on U: $\eta_i = X_{e_i}$. $\eta_i \in D \Rightarrow \forall \zeta \in D : \omega(\eta_i, \zeta) = 0 \Rightarrow \forall \zeta \in D : \zeta e_i = 0 \Rightarrow \exists \hat{e}_i$ functions on M/D: $e_i = \hat{e}_i \circ \pi = \pi^* \hat{e}_i$. If (c_i) is the basis of $T^*_x(M/D)$ dual to (ξ_i) then we have:

$$
\begin{aligned}
& (\tilde{\xi}) = (\eta|_{m_o}) \\
\Longleftrightarrow\ & i_{\eta_j|m_o} \omega + \pi^*_{m_o} c_j = 0 \\
\Longleftrightarrow\ & \pi^*_{m_o} c_j = de_j \qquad (i_{X_j} \omega + df = 0 \text{ by definition}) \\
& \qquad = d(\pi^* \hat{e}_j) \\
& \qquad = \pi^*_{m_o} d\hat{e}_j \\
\Rightarrow\ & \pi^*_m c_j = \pi^*_m d\hat{e}_j \qquad \forall m \in U \cap \pi^{-1}(x) \\
\Longleftrightarrow\ & (\tilde{\xi}) = (\eta|_m). \quad \boxed{\text{QED}}
\end{aligned}
$$

Let us turn back to ½-densities on M/D: a ½-density ϕ on M/D is a complex function on $F^n(M/D)$ enjoying certain properties. Using the above constructed map, ϕ defines a function $\tilde{\phi}$ on R as follows: suppose $(\eta_i) \in R_m$, i.e. (η_i) is a frame of D_m, then there exists a unique frame $(\xi_i) \in F^n_{\pi(m)}(M/D)$ such that $(\tilde{\xi}) = (\eta)$ and we define:

$$\tilde{\phi}((\eta)) \equiv \tilde{\phi}((\tilde{\xi})) = \phi((\xi)).$$

If ϕ is a $\frac{1}{2}$-density and $g \in GL(n,\mathbb{R})$ then

$$\phi(\xi g) = \phi(\xi) \cdot |\det g|^{\frac{1}{2}}$$

hence for $\tilde{\phi}$ we have:

$$\tilde{\phi}(\tilde{\xi}g) = \tilde{\phi}(\widetilde{\xi(g^{-1})^T}) = \phi(\xi(g^{-1})^T)$$

$$= \phi(\xi)\ |\det (g^{-1})^T|^{\frac{1}{2}}$$

$$= \tilde{\phi}(\tilde{\xi})\ |\det g|^{-\frac{1}{2}} .$$

We see that the lift $\tilde{\phi}$ of a $\frac{1}{2}$-density ϕ is a function on R satisfying a certain relation; we now define a $-\frac{1}{2}$-D-density ν as a function on R such that for $\eta \in R_m$, $g \in GL(n,\mathbb{R})$:

$$\nu(\eta g) = \eta(\nu)\ |\det g|^{-\frac{1}{2}}.$$

(N.B. in [Simms & Woodhouse] such a function is called a $+\frac{1}{2}$-D-density!).

In the same way as we defined the bundle $\Delta^{\frac{1}{2}}X$ from F^nX (i.e. a section of $\Delta^{\frac{1}{2}}X$ $\equiv$ a $\frac{1}{2}$-density on X $\equiv$ a function on F^nX), we now define B^D to be the complex line-bundle whose fibres B^D_m consist of all functions $\nu_m : R_m \to \mathbb{C}$ such that for $\eta \in R_m$, $g \in GL(n,\mathbb{R})$:

$$\nu_m(\eta g) = \nu_m(\eta)\ |\det g|^{-\frac{1}{2}} .$$

With this definition a section ν of B^D represents a $-\frac{1}{2}$-D-density on M, i.e. ν represents a function on R satisfying the $-\frac{1}{2}$-D-density condition.

The following question now arises: each $\frac{1}{2}$-density ϕ on M/D can be lifted to a $-\frac{1}{2}$-D-density $\tilde{\phi}$ on R, in other words, each section ϕ of $\Delta^{\frac{1}{2}}(M/D)$ can be lifted to a section $\tilde{\phi}$ of B^D but ... is each section of B^D the lift of a $\frac{1}{2}$-density, and if not, how do we describe the subset of sections of B^D which are the lift of $\frac{1}{2}$-densities?

It should be "obvious" that not all sections ν of B^D are the lift of a section ϕ of $\Delta^{\frac{1}{2}}(M/D)$ because of the following: if ϕ is a ½-density then the variation of $\tilde{\phi}$ on a leaf of the foliation D (which is projected to one point in M/D) is determined completely by the value in one point, or equivalently by the value of ϕ at the projection. In a certain sense $\tilde{\phi}$ is constant along the leaves of D, so it remains to specify in what sense.

Let us investigate this situation in local canonical coordinates (q,p) on a neighbourhood U in which D is spanned by $\left\{\frac{\partial}{\partial p_i}\right\}$. If ϕ is a ½-density on M/D then we have seen that $\tilde{\phi}$ is defined by

$$\tilde{\phi}(m, (\eta_i)) = |\det g(m)|^{-\frac{1}{2}} \tilde{\phi}(m, (X_{q^i})) = $$
$$= |\det g(m)|^{-\frac{1}{2}} \phi(\pi(m), (\frac{\partial}{\partial q^i}))$$

if $$\eta_i = -\frac{\partial}{\partial p_j}\Big|_m g_{ji}(m) = X_{q^i}\Big|_m g_{ji}(m).$$

Using the same local coordinates we can identify (locally: on U) -½-D-densities ν with functions $\hat{\nu}$ on M by

$$\hat{\nu}(m) = \nu(m, (X_{q^i}))$$
$$\Rightarrow \nu(m, (\eta_i)) = |\det g(m)|^{-\frac{1}{2}} \nu(m)$$

and we see that the -½-D-density $\tilde{\phi}$ is constant along the leaves of D in the sense that the function $\hat{\tilde{\phi}}$ is a function constant along the leaves of D or in other "words":

$$\forall \zeta \in D : \zeta\hat{\tilde{\phi}} = \zeta\tilde{\phi}(m, (X_{q^i}\big|_m)) = 0 \qquad \text{(as a function on U).}$$

We are now faced with several problems/questions:

1°. an aesthetical question: can we find a way to express this property of $\tilde{\phi}$ in a coordinate independent way ? ; 2°. is this condition really coordinate independent?. 3°. a more important question: is the converse true: is every $-\frac{1}{2}$-D-density ν on M satisfying

$$\forall \zeta \in D : \zeta\hat{\nu} = 0$$

the lift of a $\frac{1}{2}$-density ϕ on M/D ?

We start with the first two questions simultaneously: the (local) coordinates comes in when we write $\nu(m, (X_{q^i}))$; we can do "better" and substitute arbitrary vector fields (spanning D !) instead of the (X_{q^i}) but nevertheless, in order to write $\zeta\hat{\nu} = 0$ we have to substitute some vector fields which implies an arbitrary choice. Instead of degrading a $-\frac{1}{2}$-D-density ν to a function on M (by substituting vector fields) we can promote vector fields $\zeta \in D$ to operate on $-\frac{1}{2}$-D-densities: suppose ν is a section of B^D, i.e. ν is a $-\frac{1}{2}$-D-density, and suppose $\zeta \in D$ is a vector field on M, then we define the section $\nabla_\zeta \nu$ of B^D as follows:

$$(\nabla_\zeta \nu)(m_o, (\eta_o)) = \zeta_{m_o} \nu(m, (\eta_m))$$

where (η_o) is an arbitrary frame of D_{m_o} and where $\{\eta_i\}$ are n linear independent locally hamiltonian vector fields on M such that $(\eta|_{m_o}) = (\eta_o)$ and such that $(\eta_i)|_m$ span D_m in a neighbourhood of m_o (such vector fields exist because (again) D is a polarization).

One should not be surprised at the use of locally hamiltonian vector fields spanning D in the definition of ∇_ζ: they already appeared in the definition of $\hat{\nu}$ (although very special ones) but they also appeared in the lift $\tilde{\xi}$ of a vector field ξ on M/D as the dependence of $\tilde{\xi}$ along a leaf of D ! The following proposition justifies our terminology:

<u>PROPOSITION</u>: *if* ν *is a* $-\frac{1}{2}$-D-*density then* $\nabla_\zeta \nu$ *is correctly defined (i.e. independent of the choice of* (η_i)*); moreover* $\nabla_\zeta \nu$ *is a* $-\frac{1}{2}$-D-*density.*

PROOF: let (η_o') be a different frame of D_{m_o}, $g \in GL(n,\mathbb{R})$ such that $(\eta_o') = (\eta_o)g$ and let (η_i') be a set of n independent locally hamiltonian vector fields spanning D such that $(\eta_i')\big|_{m_o} = (\eta_o')$. We then have to prove that:

$$(\nabla_\zeta \nu)(m_o, (\eta_o)g) = (\nabla_\zeta \nu)(m_o, (\eta_o))\,|\det g|^{-\frac{1}{2}}$$

$$\Longleftrightarrow \zeta_{m_o} \nu(m, (\eta_i')\big|_m) = |\det g|^{-\frac{1}{2}} \quad \zeta_{m_o} \nu(m, (\eta_i)\big|_m).$$

Define $g(m) \in GL(n,\mathbb{R})$ by $(\eta_i')\big|_m = (\eta_i)\big|_m g(m)$, i.e. $\eta_i' = \eta_j g(m)_{ji}$, define local functions e_i and e_i' by $\eta_i = X_{e_i}$, $\eta_i' = X_{e_i'}$ (η and η' are locally hamiltonian) and let $X_f \in D$ be an arbitrary locally hamiltonian vector field, then:

$$[X_f, X_{e_i'}] = X_{\omega(X_f, X_{e_i'})} = 0 \qquad \text{(D is isotropic!)}$$

$$\Rightarrow \; 0 = [X_f, g_{ji} X_{e_j}] = g_{ji}[X_f, X_{e_j}] + (X_f\, g_{ji}) X_{e_j}$$

$$= (X_f\, g_{ji}) X_{e_j} \qquad \text{(again because D is isotropic)}$$

Since the $X_{e_j} = \eta_j$ are independent it follows that $X_f\, g_{ji} = 0$, hence (because the locally hamiltonian vector fields in D span D by definition) we have:

$$\forall \zeta \in D_m : \zeta g_{ji} = 0.$$

Finally we can prove what we had to prove: if $\zeta \in D$ then:

$$\zeta_{m_o} \nu(m, (\eta_i')\big|_m) = \zeta_{m_o} \nu(m, (\eta_i)\big|_m g(m))$$

$$= \zeta_{m_o} \{ |\det g(m)|^{-\frac{1}{2}} \nu(m, (\eta_i)\big|_m) \}$$

$$= |\det g(m_o)|^{-\frac{1}{2}} \zeta_{m_o} (m, (\eta_i)\big|_m). \qquad \boxed{\text{QED}}$$

We continue with some properties of the map ∇_ζ :

PROPOSITION: *the map* ∇_ζ *possesses all properties of a flat (partial) connection: if* $\zeta,\eta \in D$, *if* f,g *are functions on* M *and if* ν *and* μ *are* $-\frac{1}{2}$-D-*densities then:*

$$\nabla_{f\zeta+g\eta}\nu = f\nabla_\zeta\nu + g\nabla_\eta\nu$$

$$\nabla_\zeta(\nu+\mu) = \nabla_\zeta\nu + \nabla_\zeta\mu$$

$$\nabla_\zeta f\nu = f\nabla_\zeta\nu + (\zeta f)\nu$$

$$\nabla_\zeta\nabla_\eta\nu - \nabla_\eta\nabla_\zeta\nu = \nabla_{[\zeta,\eta]}\nu \ .$$

PROOF: a direct consequence of the definition of $\nabla_\zeta\nu$. QED

REMARK: ∇_ζ is called a partial connection because it is defined for a restricted class of vector fields only.

Finally, we are going to state (and prove) the desired proposition concerning the lift of $\frac{1}{2}$-densities:

PROPOSITION: *let* ν *be a* $-\frac{1}{2}$-D-*density on* M *then: there exists a* $\frac{1}{2}$-*density* ϕ *on* M/D *such that* $\tilde{\phi} = \nu$ *if and only if* $\forall\zeta \in D : \nabla_\zeta\nu = 0$.

PROOF: [only if]: $(\nabla_\zeta\tilde{\phi})(m_o, (\eta_o)) = \zeta_{m_o}\tilde{\phi}(m, (\eta)|_m)$

$$= \zeta_{m_o}\phi(\pi(m), (\xi(m)))$$

where $(\xi(m))$ is a frame at $\pi(m)$ such that $(\xi(m)) = (\eta)_m$. But, we have seen that if (η) is a locally hamiltonian frame then $(\xi(m))$ does depend on $\pi(m)$ only hence

$$(\nabla_\zeta\tilde{\phi})(m_o, (\eta_o)) = \zeta_{m_o}\phi(\pi(m), (\xi(\pi(m)))) = 0$$

because $\pi_*\zeta_{m_o} = 0 \quad (\zeta \in D)$.

[if]: define ϕ by $\phi(x, (\xi)) = \nu(m, (\eta_m))$ where $\pi(m) = x$, $(\tilde{\xi}) = (\eta_m)$, then ϕ obviously is a ½-density satisfying the condition $\tilde{\phi} = \nu$, provided that $\nu(m, (\eta_m))$ does not depend on $m \in \pi^{-1}(x)$ (under the condition $(\eta_m) = (\tilde{\xi})$).

To prove this, let $m_o \in \pi^{-1}(x)$ be arbitrarily chosen and let (η_i) be n independent locally hamiltonian vector fields such that $\{\eta_i\}$ span D in a neighbourhood U of m_o and such that $(\eta_i|_{m_o}) = (\tilde{\xi})$. Then $(\eta_i|_m) = (\tilde{\xi})$ on U according to an earlier proposition and we have to prove that $\nu(m, \eta_m))$ is constant on $\pi^{-1}(x)$. Let $\zeta \in D$ be an arbitrary vector field on U then

$$\zeta_m \nu(m', (\eta|_{m'})) = (\nabla_\zeta \nu)(m, (\eta_m)) = 0.$$

Since ζ is arbitrary and since D "is" the tangent space of the leaves $\pi^{-1}(x)$, $\nu(m, (\eta_m))$ is constant on the leaves of D. QED

For those who have forgotten our aim, let us summarize: in order to bring ½-densities on M/D and sections of L on the same footing, we constructed the bundle R (depending on the real polarization D). Then we defined $-\frac{1}{2}$-D-densities on M as a special class of functions on R and we defined the bundle B^D as the line-bundle whose sections are the $-\frac{1}{2}$-D-densities on M. We showed that ½-densities on M/D, i.e. sections of $\Delta^{\frac{1}{2}}(M/D)$ could be lifted to sections of B^D, i.e. $-\frac{1}{2}$-D-densities on M and we introduced a partial connection ∇ on B^D in order to prove a desired result: every lift of a ½-density on M/D is covariantly constant along D (using this connection) and vice-versa: every $-\frac{1}{2}$-D-density covariantly constant along D is the lift of a ½-density on M/D.

6 QUANTIZATION I

In the previous section we defined the bundle B^D in order to bring $\frac{1}{2}$-densities on M/D and sections of L on the same footing. We now define the quantum bundle $QB = L \otimes B^D$ and we want to define an inner product on the sections of this bundle according to the heuristic formula

$$< s_1 \otimes \nu_1 , s_2 \otimes \nu_2 > = \int_{M/D} (s_1, s_2) \bar{\phi}_1 \phi_2$$

where $\nu_i = \tilde{\phi}_i$, the lift of a $\frac{1}{2}$-density on M/D. The first idea is to restrict our attention to those sections ψ of QB which admit a local representation $\psi = s \otimes \nu$ where s and ν are covariantly constant along D (according to the (partial) connections ∇ on L and B^D), but ... how do we see whether ψ admits such a representation or not? To "answer" this question we define a partial connection ∇ on QB as follows: for $\zeta \in D$, ψ a section of QB with local representation $\psi = s \otimes \nu$ we define

$$\nabla_\zeta \psi = (\nabla_\zeta s) \otimes \nu + s \otimes (\nabla_\zeta \nu) .$$

This is a correct definition, independent of the local representation $s \otimes \nu$ just because the ∇ are connections in L and B^D: if f is a function on M then $f(s \otimes \nu) = (fs) \otimes \nu = s \otimes (f\nu)$ and $(\nabla_\zeta (fs)) \otimes \nu + (fs) \otimes (\nabla_\zeta \nu) =$ $= (\nabla_\zeta s) \otimes (f\nu) + s \otimes (\nabla_\zeta (f\nu)) = (\zeta f) s \otimes \nu + f \nabla_\zeta (s \otimes \nu)$.

<u>PROPOSITION</u>: $\mathrm{curv}(\nabla) = \frac{1}{\hbar}\omega$.

<u>PROOF</u>: a direct consequence of the fact that $\mathrm{curv}_L(\nabla) = \omega/\hbar$ and $\mathrm{curv}_{B^D}(\nabla) = 0$. QED

<u>PROPOSITION</u>: *if* ψ *admits a local representation* $\psi = s \otimes \nu$ *where* s *and* ν *are covariantly constant along* D *then* ψ *is covariantly constant along* D.

PROOF: $\nabla_\zeta \psi = \nabla_\zeta (s \otimes \nu) = (\nabla_\zeta s) \otimes \nu + s \otimes (\nabla_\zeta \nu) = 0 + 0 = 0.$ QED

It seems that we need the converse of the proposition above in order to characterize the sections of QB which admit the desired representation, but fortunately we do not need such a proposition. Our aim was and is to define an inner product on sections of QB by associating a density on M/D to a pair of sections; we know how to do it for sections which admit a special representation ($\psi = s \otimes \nu$ where s and ν are both covariant constant along D) and we will show how to do it for sections ψ which are covariant constant along D, with or without a special representation. Although we will in this section construct explicitly a local representation $\psi = s \otimes \nu_o$ in which both s and ν_o are covariant constant along D if ψ is; in a more general case however we will not use such representations (see section 9).

Our first step is to give a short procedure to compute the density $\bar{\phi}_1 \phi_2$ on M/D if we know the $-\frac{1}{2}$-D-densities $\tilde{\phi}_1$ and $\tilde{\phi}_2$ on M. Suppose $x \in M/D$, $m \in \pi^{-1}(x)$, $(\zeta_i) = (\zeta_1, \ldots, \zeta_n)$ a basis of D_m and $\xi_i \in T_m M$ such that $(\zeta_1, \ldots, \zeta_n, \xi_1, \ldots, \xi_n)$ is a basis for $T_m M$, then $(\pi_* \xi_i)$ is a basis for $T_x(M/D)$ because π is a submersion and for $\frac{1}{2}$-densities ϕ_1 and ϕ_2 on M/D we have the following lemma:

LEMMA: $\overline{\tilde{\phi}_1(\xi)} \cdot \tilde{\phi}_2(\xi) \cdot |\varepsilon_\omega(\zeta_1, \ldots, \zeta_n, \xi_1, \ldots, \xi_n)| = \overline{\phi_1(\pi_* \xi)} \cdot \phi_2(\pi_* \xi).$

PROOF: we use local canonical coordinates (q,p) such that D is spanned by $\left\{X_{q^i}\right\} = \left\{-\frac{\partial}{\partial p_i}\right\}$; then (q) are coordinates on M/D and there exist matrices $g \in GL(n, \mathbb{R})$, $h \in GL(2n, \mathbb{R})$ such that:

$$(\zeta, \xi) = \left(-\frac{\partial}{\partial p_1}, \ldots, -\frac{\partial}{\partial p_n}, \frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n}\right) \cdot h$$

and $(\pi_*\xi) = \left(\frac{\partial}{\partial q^1},\ldots,\frac{\partial}{\partial q^n}\right)\cdot g$ hence $h = \begin{pmatrix} a & b \\ \emptyset & g \end{pmatrix}$ where $a \in GL(n,\mathbb{R})$.

Now $\varepsilon_\omega\left(\frac{\partial}{\partial p_1},\ldots,\frac{\partial}{\partial p_n}, \frac{\partial}{\partial q^1},\ldots,\frac{\partial}{\partial q^n}\right) = 1$ because $\varepsilon_\omega = \text{const}\cdot\omega^n =$

$= dp_1 \wedge \ldots \wedge dp_n \wedge dq^1 \wedge \ldots \wedge dq^n$, and $\tilde{\phi}_j(\zeta) = \tilde{\phi}_j\left(-\frac{\partial}{\partial p_i}\right)\cdot |\det a|^{-\frac{1}{2}} =$

$= \phi_j\left(\frac{\partial}{\partial q^i}\right)\cdot |\det a|^{-\frac{1}{2}}$ hence $\overline{\tilde{\phi}_1(\zeta)}\cdot\tilde{\phi}_2(\zeta)\cdot|\varepsilon_\omega(\zeta,\xi)| =$

$= |\det a|^{-1}\,\overline{\phi_1\left(\frac{\partial}{\partial q}\right)}\cdot\phi_2\left(\frac{\partial}{\partial q}\right)\cdot|\det h| = \overline{\phi_1\left(\frac{\partial}{\partial q}\right)}\cdot\phi_2\left(\frac{\partial}{\partial q}\right)\cdot|\det g| =$

$= \overline{\phi_1(\pi_*\xi)}\cdot\phi_2(\pi_*\xi)$. QED

The proof of this lemma can be generalized immediately to give the following proposition.

PROPOSITION: *using the same notation as above, two sections* $\psi_1 = s_1\otimes\nu_1$ *and* $\psi_2 = s_2\otimes\nu_2$ *of* QB *define a density* ϕ_m *at* $x = \pi(m) \in M/D$ *for each* m *by*

$$\phi_m(\pi_*\xi) = (s_1,s_2)(m)\overline{\nu_1(\zeta)}\cdot\nu_2(\zeta)\cdot|\varepsilon_\omega(\zeta,\xi)|$$

The definition of ϕ_m does not depend upon the (local) representation $\psi_i = s_i\otimes\nu_i$ but ϕ_m depends in general upon the choice of $m \in \pi^{-1}(x)$. However, for ψ_i covariant constant along D it does not depend upon $m \in \pi^{-1}(x)$.

PROPOSITION: *if* ψ_1 *and* ψ_2 *are covariant constant along* D *then* ϕ_m *does not depend upon the choice of* $m \in \pi^{-1}(x)$.

PROOF: we show that ϕ_m is locally independent of m ; from the connectedness of $\pi^{-1}(x)$ (= a leaf of D) the proposition follows. As before we use local canonical coordinates (q,p) defined on a neighbourhood U of $m_o \in \pi^{-1}(x)$ such that D is spanned by $\{X_{q^i}\}$ on U. On U we have a symplectic potential $\theta = p_i\,dq^i$, $d\theta = \omega$ which defines a (local) trivialization of L such that sections s of L over U can be identified with complex functions s on U and such that the connection ∇ is defined by $\nabla_\xi s = \xi s - \frac{i}{\hbar}\theta(\xi)s$.

A priori it is not clear that such an arbitrary choice of θ leads to a correct description, because L was constructed from a special set of local potentials θ_i. However, if we restrict U to be contained in such a set and if we use inside this set (which defined the local trivialization of L) a gauge transformation, then it follows that our construction above is indeed correct!

Because D is spanned by $\left\{\frac{\partial}{\partial p_i}\right\}$ it follows that

$$\zeta \in D \Rightarrow \nabla_\zeta s = \zeta s \qquad \text{on } U.$$

Furthermore, on U we define a local trivialization of B^D by means of the $-\frac{1}{2}$-D-density ν_o defined by

$$\nu_o(m, (X_{q^i})) = 1.$$

Then each $-\frac{1}{2}$-D-density ν on U defines a unique complex function f on U such that

$$\nu(m, \left(-\frac{\partial}{\partial p_i} g_{ij}\right)) = f(m)\ |\det g|^{-\frac{1}{2}}.$$

If we compute $\nabla_\zeta \nu_o$ (defined for $\zeta \in D$ only) then:

$$(\nabla_\zeta \nu_o)(m_o, X_{q^i}\big|_{m_o}) = \zeta_{m_o}\ \nu_o(m, X_{q^i}\big|_m) = 0$$

hence ν_o is covariant constant along D. Now if $\psi = s \otimes \nu$ is any section of QB and if $\zeta \in D$ then there exists $f\colon U \to \mathbb{C}$ such that $\nu = f\nu_o$ and:

$$\nabla_\zeta \psi = (\nabla_\zeta s) \otimes \nu + s \otimes \nabla_\zeta \nu = (\zeta s) \otimes (f\nu_o) + s \otimes (\zeta f)\nu_o =$$
$$= (f\zeta s + s\zeta f) \otimes \nu_o = (\zeta(fs)) \otimes \nu_o .$$

Hence ψ is covariant constant along D iff the function fs is independent of the coordinates p_i, i.e. iff fs is a function on M/D. Now if ψ_1 and ψ_2 are covariant constant along D then

$$\phi_m\left(\frac{\partial}{\partial q^i}\right) = \bar{s}_1(m)\, s_2(m)\, \overline{\nu_1(X_{q^i})}\, \nu_2(X_{q^i}) \cdot \left|\varepsilon_\omega\left(X_{q^i}, \frac{\partial}{\partial q^i}\right)\right|$$
$$= \overline{s_1(m)}\, s_2(m)\, \overline{f_1(m)}\, f_2(m) = \overline{(s_1 f_1)(m)}(s_2 f_2)(m)$$

which is independent of the coordinates p_i or equivalently which does not depend upon the choice of $m \in \pi^{-1}(x) \cap U$. QED

We conclude that we can define unambiguously a density (ψ_1,ψ_2) on M/D out of two sections ψ_1 and ψ_2 of QB which are covariant constant along D by means of the formula

$$(\psi_1,\psi_2) = (s_1,s_2)\, \bar{\nu}_1 \nu_2 |\varepsilon_\omega|$$
$$\iff (\psi_1,\psi_2)(\pi(m), \pi_* \xi) =$$
$$(s_1,s_2)(m) \cdot \overline{\nu_1(m,\zeta)} \cdot \nu_2(m,\zeta)\, |\varepsilon_\omega(\zeta,\xi)|$$

where (ζ,ξ) a basis of $T_m M$ and (ζ) a basis of D_m.

We now define the set PH as the set of C^∞-sections ψ of QB which are covariant constant along D and for which $\int_{M/D} (\psi,\psi)$ is finite:

$$PH = \{\psi : M \to QB \mid \forall \zeta \in D : \nabla_\zeta \psi = 0 \wedge \int_{M/D} (\psi,\psi) < \infty\}.$$

On PH we define an inner product $<\,,\,>$ as follows:

$$<\psi_1,\psi_2> = \int_{M/D} (\psi_1,\psi_2)$$

which transforms PH into a prehilbert space. Finally, we define the Hilbert space H as the completion of PH and we interpret H as representing the Hilbert space of the quantum mechanical description.

Before we turn our attention to the question of observables, we make a few remarks: the bundle QB is a complex line-bundle over M with a partial connection ∇ (where $\mathrm{curv}(\nabla) = \omega/\hbar$) and an "inner product" in the fibres (which defines a density in $\pi(m) \in M/D$). We see that QB resembles L in many ways: L is a complex line-bundle over M with a connection ∇ (where $\mathrm{curv}(\nabla) = \omega/\hbar$) and an inner product in the fibres (which defines a density on M by means of the natural volume element ε_ω on M).

What we have done in fact is that we have added some structure to L such that the inner product in the fibres defines a density on M/D (after the introduction of a real polarization) instead of a density on M. We needed this trick to reduce the size of the Hilbert space: first sections of L, then sections of L covariant constant along D (which defined the trivial Hilbert space $\{0\}$) and finally sections of QB covariant constant along D. The reason for this construction was that we wanted to imitate the Schrödinger quantization as best we can.

We have seen already (in a heuristical way) that this approach leads us to the Hilbert space of the Schrödinger quantization when we use $D = D^V$, so it now remains to specify the observables which are quantizable, i.e. we have to define a map δ: observables $\to$ self-adjoint operators on H and we have to define the domain of δ.

However, before we can define δ, we need some additional techniques: suppose ζ is a vector field on M with flow ρ_t, i.e. $\left.\frac{d}{dt}\right|_{t=0} \rho_t(m) = \zeta_m$, then ρ_{t*} is the associated map on tangent vectors which defines a flow $\tilde{\rho}_t$ in F^nM:

$$\tilde{\rho}_t(m, (\eta_i)) := (\rho_t m, (\rho_{t*}\eta_i)).$$

When we want to restrict $\tilde{\rho}_t$ to the subbundle R of F^nM (R = the bundle of all D-frames) then R should be invariant under $\tilde{\rho}_t$, which imposes some restrictions on the vector field ζ. If we abbreviate the expression "$\forall \eta \in D : [\zeta,\eta] \in D$" by $[\zeta,D] \subset D$ then one can prove that the following condition is sufficient to guarantee that R is invariant under $\tilde{\rho}_t$.

<u>LEMMA</u>: suppose $[\zeta,D] \subset D$ and $(m,\eta) \in R$ then $\tilde{\rho}_t(m,\eta) \in R$ (i.e. $\tilde{\rho}_t$ can be restricted to R) and furthermore there exists a flow σ_t on M/D such that $\pi \circ \rho_t = \sigma_t \circ \pi$ (i.e. ρ_t projects to a flow σ_t on M/D).

With the aid of the flow $\tilde{\rho}_t$ we define (for vector fields ζ satisfying $[\zeta,D] \subset D$) the lift of ζ as the vector field $\tilde{\zeta}$ on R given by:

$$\tilde{\zeta}(m,\eta)' := \frac{d}{dt}\Big|_{t=0} \tilde{\rho}_t(m,\eta)$$

and one can prove that if both ζ_1 and ζ_2 can be lifted to R (i.e. $[\zeta_1,D] \subset D$ and $[\zeta_2,D] \subset D$) then:

$$\widetilde{[\zeta_1,\zeta_2]} = [\tilde{\zeta}_1,\tilde{\zeta}_2] .$$

Now if ν is a $-\frac{1}{2}$-D-density then it is a function on R hence if ζ satisfies $[\zeta,D] \subset D$ then we can apply $\tilde{\zeta}$ and $\tilde{\zeta}\nu$ is again a function on R. In the literature the function $\tilde{\zeta}\nu$ is often denoted by $L_\zeta\nu$ and the operator L_ζ "conserves" $-\frac{1}{2}$-D-densities:

<u>PROPOSITION</u>: *if* ν *is a* $-\frac{1}{2}$-D-*density, then* $L_\zeta\nu$ *too.*

<u>PROOF</u>:
$$\begin{aligned}(\tilde{\zeta}\nu)(m, (\eta)g) &= \frac{d}{dt}\Big|_{t=0} \nu(\rho_t m, \rho_{t*}(\eta_i)g) = \\ &= \frac{d}{dt}\Big|_{t=0} \nu(\rho_t m, (\rho_{t*}\eta_i)g) = \\ &= \frac{d}{dt}\Big|_{t=0} \nu(\rho_t m, (\rho_{t*}\eta_i)) \cdot |\det g|^{-\frac{1}{2}} \\ &= (\tilde{\zeta}\nu)(m, (\eta)) \cdot |\det g|^{-\frac{1}{2}} .\end{aligned}$$
QED

In general $\tilde{\zeta}(m,\eta)$ will depend on the values of ζ in a neighbourhood of m and not only of ζ_m, but for special vector fields we have a nice property:

<u>PROPOSITION</u>: *let* ζ *be a locally hamiltonian vector field,* $\zeta \in D$ *and let* ν *be a* $-\frac{1}{2}$-D-*density on* M *then:*

$$L_\zeta \nu = \tilde{\zeta}\nu = \nabla_\zeta \nu.$$

<u>PROOF</u>: $(\nabla_\zeta \nu)(m_o,\eta_o) = \zeta_{m_o} \nu(m,\eta_m) = \frac{d}{dt}\Big|_{t=0} \nu(\rho_t m_o, (\eta_i)\big|_{\rho_t m_o})$. Now ζ and η_i are locally hamiltonian vector fields hence there exist functions z, e_i on M such that $\zeta = X_z$, $\eta_i = X_{e_i}$ so $[\zeta,\eta_i] = X_{[z,e_i]}$. But $[z,e_i] = \omega(\zeta,\eta_i) = 0$ because $\zeta,\eta_i \in D$ hence $[\zeta,\eta_i] = 0 \Rightarrow \rho_{t*}\eta_i\big|_{m_o} = \eta_i\big|_{\rho_t m_o}$.

Finally $(\tilde{\zeta}\nu)(m_o,\eta_o) = \frac{d}{dt}\Big|_{t=0} \nu(\rho_t m_o, (\rho_{t*}\eta_i))$

$$= \frac{d}{dt}\Big|_{t=0} \nu(\rho_t m_o, (\eta_i\big|_{\rho_t m_o})) = (\nabla_\zeta \nu)(m_o,\eta_o). \quad \boxed{\text{QED}}$$

With these technical resources we can define the map δ and its domain: a function $f: M \to \mathbb{R}$ is called a quantizable observable if the associated globally hamiltonian vector field X_f satisfies the condition $[X_f,D] \subset D$. For quantizable observables we define the operator $\delta(f)$ on sections $\psi = s \otimes \nu$ of QB as follows:

$$\delta(f)\psi = (-i\hbar \nabla_{X_f} s + fs) \otimes \nu - i\hbar\, s \otimes (L_{X_f} \nu)$$

which reduces to $\delta(f)\psi = -i\hbar \nabla_{X_f} \psi + f\psi$ if $X_f \in D$.

Several questions now have to be answered: 1. is $\delta(f)$ an operator on H, i.e. is $\delta(f)\psi$ an element of H if ψ is?; 2. is $\delta(f)$ self-adjoint on H? To answer these questions, we first state a proposition.

PROPOSITION: *if* f *is a quantizable observable, if* $\zeta \in D$ *is locally hamiltonian and if* ψ *is a section of* QB *then:*

$$\nabla_\zeta \delta(f)\psi = \delta(f)\nabla_\zeta\psi - i\hbar\, \nabla_{[\zeta, X_f]}\psi \,.$$

PROOF: a direct consequence of the definitions if one uses the following facts:

a) $\mathrm{curv}(\nabla) = \omega/\hbar$ in L

b) $\nabla_\zeta \nu = L_\zeta \nu$

c) $L_\zeta L_{X_f} \nu = L_{X_f} L_\zeta \nu + L_{[\zeta, X_f]}\nu$ (because $\widetilde{[\zeta, X_f]} = [\tilde{\zeta}, \tilde{X}_f]$)

d) $L_{[\zeta, X_f]}\nu = \nabla_{[\zeta, X_f]}\nu$ (because $[\zeta, X_f]$ is globally hamiltonian in D).

QED

COROLLARY: *if* ψ *is covariant constant along* D *then* $\delta(f)\psi$ *too.*

PROOF: D is (locally) spanned by locally hamiltonian vector fields and we apply the proposition above. QED

We now have proved that $\delta(f)$ maps sections which are covariant constant along D into sections covariant constant along D, so there remains to show that $\delta(f)$ maps (a part of) H into H and that $\delta(f)$ is (essentially) self-adjoint on H. As in previous sections we will "show" that if f is a quantizable observable such that X_f is complete then $\delta(f)$ is essentially self-adjoint on H.

If X_f is a complete vector field, then there exists a 1-parameter group of transformations $\hat{\rho}_t$ on the sections of L such that:

$$(\hat{\rho}_t s_1, \hat{\rho}_t s_2)(m) = (s_1, s_2)(\rho_t m)$$

$$-i\hbar \frac{d}{dt}\Big|_{t=0} \rho_t s = -i\hbar\, \nabla_{X_f} s + fs \,,$$

a fact which was proved heuristically in section 2. Furthermore, the 1-parameter group ρ_t on M associated to X_f induces the 1-parameter group $\tilde{\rho}^*_t$ on B^D such that for ν a $-\frac{1}{2}$-D-density:

$$\frac{d}{dt}\Big|_{t=0} \tilde{\rho}^*_t \nu = \tilde{X}_f \nu = L_{X_f} \nu,$$

where $\tilde{\rho}^*_t \nu$ is the pullback of $\nu : \tilde{\rho}^*_t \nu = \nu \circ \tilde{\rho}_t$ and where ρ_t is the flow induced on R. We now can combine these two flows to define a flow $\check{\rho}_t$ on QB:

$$\check{\rho}_t \psi = \check{\rho}_t (s \otimes \nu) = (\hat{\rho}_t s) \otimes (\tilde{\rho}^*_t \nu)$$

which is defined correctly, independent of the representation $\psi = s \otimes \nu$ because $\hat{\rho}_t$ and $\tilde{\rho}^*_t$ are linear over the functions on M.

<u>PROPOSITION</u>: $\check{\rho}_t$ *is a unitary flow on* H.

<u>PROOF</u>: suppose $\zeta \in D_{m_o}$ then for a $-\frac{1}{2}$-D-density ν we have:

$$(\nabla_\zeta \tilde{\rho}^*_t \nu)(m_o, \eta_o) = \zeta_{m_o} \nu(\rho_t m, \rho_{t*} \eta_m),$$

but since ρ_t leaves ω invariant we have that η locally hamiltonian $\Rightarrow \rho_{t*}\eta$ locally hamiltonian hence $\rho_{t*}\eta_m = \xi_{\rho_t m}$ for some locally hamiltonian frame ξ. Now we can compute:

$$\begin{aligned}(\nabla_\zeta \tilde{\rho}^*_t \nu)(m_o, \eta_o) &= (\rho_{t*} \xi_{m_o}) \nu(m, \xi_m) \\ &= (\nabla_{\rho_{t*}\zeta_{m_o}} \nu)(\rho_t m_o, \rho_{t*}\eta_o) \\ &= (\tilde{\rho}^*_t \nabla_{\rho_{t*}\zeta} \nu)(m_o, \eta_o)\end{aligned}$$

hence $\nabla_\zeta \tilde{\rho}^*_t \nu = \tilde{\rho}^*_t \nabla_{\rho_{t*}\zeta} \nu$.

Using the techniques of the last part of section 2 one can show that for a section s of L:

$$\nabla_\zeta \hat{\rho}_t s = \rho_t \nabla_{\rho_{t*}\zeta} s .$$

With these facts it follows that if $\psi = s \otimes \nu$ is covariant constant along D then $\check{\rho}_t \psi$ also. We now suppose $\psi_1 = s_1 \otimes \nu$ and $\psi_2 = s_2 \otimes \nu$ are two sections of QB covariant constant along D; then they define a density $\phi = (\psi_1, \psi_2)$ on M/D and if we apply $\check{\rho}_t$ we get a density $\check{\phi} = (\check{\rho}_t \psi_1, \check{\rho}_t \psi_2)$ on M/D defined by

$$\check{\phi}(x,\xi) = (\hat{\rho}_t s_1, \hat{\rho}_t s_2)(m)\overline{\tilde{\rho}_t^* \nu_1 (m,\eta)}\tilde{\rho}_t^* \nu_2 (m,\eta) \cdot |\varepsilon_\omega(\eta,\zeta)|$$

where $\pi m = x$, $\pi_* \zeta = \xi$ and η a frame of D_m. Since ρ_t is the flow of a (globally) hamiltonian vector field $\rho_t^* \varepsilon_\omega = \varepsilon_\omega$ (Liouville) and we have:

$$\varepsilon_\omega(\eta,\zeta) = \varepsilon_\omega(\rho_{t*}\eta, \rho_{t*}\zeta),$$

hence

$$\check{\phi}(x,\xi) = (s_1,s_2)(\rho_t m) \cdot \overline{\nu_1(\rho_t m, \rho_{t*}\eta)}\nu_2(\rho_t m, \rho_{t*}\eta) \cdot |\varepsilon_\omega(\rho_{t*}\eta, \rho_{t*}\zeta)| .$$

Because f is a quantizable observable we have an induced flow σ_t on M/D defined by $\sigma_t \circ \pi = \pi \circ \rho_t$ so

$$\check{\phi}(x,\xi) = \phi(\sigma_t x, \sigma_{t*}\xi) .$$

If we define for any density χ and any diffeomorphism σ of a n-dimensional manifold X a new density $\sigma^*\chi$ by

$$(\sigma^*\chi)(x,\xi) = \chi(\sigma x, \sigma_* \xi)$$

then one can prove (completely analogous to the case of n-forms on X) the formula

$$\int_X \sigma^*\chi = \int_{\sigma X} \chi = \int_X \chi .$$

Consequently, if $\psi_1, \psi_2 \in H$ (or more precisely if $\psi_1, \psi_2 \in PH$) then

$$< \check{\rho}_t\psi_i, \check{\rho}_t\psi_i > = \int_{M/D} (\check{\rho}_t\psi_i, \check{\rho}_t\psi_i) = \int_{M/D} \sigma_t^*(\psi_i, \psi_i)$$

$$= \int_{M/D} (\psi_i, \psi_i) = < \psi_i, \psi_i >$$

which shows that $\check{\rho}_t\psi_i \in H$ $(i = 1,2)$. Furthermore:

$$< \check{\rho}_t\psi_1, \check{\rho}_t\psi_2 > = < \psi_1, \psi_2 >$$

which shows that $\check{\rho}_t$ is unitary on H. QED

We have proved that $\check{\rho}_t$ is a unitary group on H hence we can apply Stone's theorem (modulo the fact that we did not prove all conditions (on $\check{\rho}_t$) of Stone's theorem) which tells us that the generator of $\check{\rho}_t$ is self-adjoint:

$$-i\hbar \frac{d}{dt}\Big|_{t=0} \check{\rho}_t\psi = (-i\hbar \frac{d}{dt}\Big|_{t=0} \hat{\rho}_t s) \otimes \nu + s \otimes (-i\hbar \frac{d}{dt}\Big|_{t=0} \tilde{\rho}_t^* \nu)$$

$$= (-i\hbar \nabla_{X_f} s + fs) \otimes \nu + s \otimes (-i\hbar L_{X_f} \nu) = \delta(f)\psi .$$

We can summarize this section as follows: on the quantum bundle $QB = L \otimes B^D$ we defined a connection ∇ which is essentially the "sum" of the connections on L and B^D. Then we defined an "inner product" (ψ_1, ψ_2) on sections covariant constant along D where (ψ_1, ψ_2) is a density on M/D in such a way that (ψ_1, ψ_2) is the mathematically rigorous formulation of the heuristic idea $(\psi_1, \psi_2) = (s_1, s_2)\bar{\phi}_1\phi_2$ (where $\psi_i = s_i\tilde{\phi}_i$).

Out of the sections ψ of QB covariant constant along D we constructed the Hilbert space H as $H = \{\psi \mid \forall\zeta\in D : \nabla_\zeta\psi = 0 \wedge \int_{M/D} (\psi,\psi) < \infty\}$, which we interpreted as the Hilbert space of the quantum theory of the classical mechanical system described by the symplectic manifold (M,ω). The quantizable observables f were defined by $[X_f, D] \subset D$ and for quantizable observables we defined an operator $\delta(f)$ on H; we then showed that if X_f is complete then $\delta(f)$ is (essentially) self-adjoint on H.

With these constructions we finally possess a quantization procedure: starting with a symplectic manifold (M,ω) we have constructed a Hilbert space H and a map δ which assigns to a quantizable observable f a self-adjoint operator $\delta(f)$ on H (actually we proved the self-adjointness only for "complete" quantizable observables, in general $\delta(f)$ is only skew-symmetric).

In the next section we will investigate some examples to see how this quantization procedure works; in particular we will see that the class of quantizable observables depends strongly on the choice of the real polarization D.

7 SOME EXAMPLES I

In this section we will give some examples to see how the quantization procedure described in the previous sections works. We will get an idea how the class of quantizable observables depends upon the polarization; furthermore we will see the shortcomings of this quantization procedure and its failures. Some of the problems encountered will be solved (partially) in the next sections, other problems will not be treated in these notes.

EXAMPLE 1: *Schrödinger quantization: position representation.*
Here we quantize the symplectic manifold (M,ω) where $M = \mathbb{R}^{2n}$ with coordinates $(q^1,\ldots,q^n, p_1,\ldots,p_n)$ and symplectic from $\omega = dp_i \wedge dq^i$; it represents the phase-space of a particle (system of particles) in $\mathbb{R}^n$.
The various "choices" in the quantization procedure are made as follows:
$L = M \times \mathbb{C}$ (which is in this case unique up to isomorphisms)
$D = D^v : D^v_m = \left\{ \frac{\partial}{\partial p_i}\Big|_m \right\}$
hence $M/D \cong \mathbb{R}^n$ with coordinates (q^i) and projection π: $\pi(q^i,p_j) = (q^i)$.
In more detail we have:

1A. Prequantization: L has a global non-vanishing section s_o defined by $s_o(m) = (m, 1)$ hence we can identify sections s of L with functions $\dot{s}$ on M by

$$s(m) = (m, \dot{s}(m)) \quad \text{or} \quad s = \dot{s} \cdot s_o .$$

The connection ∇ is given by the formula

$$(\nabla_\zeta s)^\cdot = \zeta\dot{s} - \frac{i}{\hbar}\theta(\zeta)\dot{s} \quad \text{with} \quad \theta = p_i dq^i ,$$

and the inner product by

$$(s_1,s_2)(m) = \overline{\dot{s}_1(m)}\,\dot{s}_2(m) .$$

With these ingredients, the Hilbert space of prequantization is given by

$\hat{H} = L^2(\mathbb{R}^{2n}, \lambda^{(2n)})$ = square-integrable functions on $\mathbb{R}^{2n}$ and, as we have seen in section 2, the observables p_j and q^j are represented on H by

$$\hat{\delta}(p_j) = -i\hbar\frac{\partial}{\partial q^j}\ , \qquad \hat{\delta}(q^j) = q^j + i\hbar\frac{\partial}{\partial p_j}\ .$$

<u>1B. The bundle B^D</u>: the bundle B^D has an obvious trivializing section ν_o defined by

$$\nu_o\left(m, \left(-\frac{\partial}{\partial p_i}\right)\right) = 1\ ;$$

each $-\frac{1}{2}$-D-density ν defines a function $\dot{\nu}$ on M by:

$$\nu\left(m, \left(-\frac{\partial}{\partial p_i}\right)\right) = \dot{\nu}(m)$$

and then

$$\nu\left(m, \left(-\frac{\partial}{\partial p_i}\right)g_{ij}\right) = \dot{\nu}(m)\ |\det g|^{-\frac{1}{2}} = \dot{\nu}(m)\,\nu_o\left(m, \left(-\frac{\partial}{\partial p_i}\right)g_{ij}\right)$$

$$\Longleftrightarrow \nu = \dot{\nu}\cdot\nu_o .$$

By means of ν_o we can identify B^D with $M\times\mathbb{C}$ and we compute the partial connection ∇ on B^D in terms of $M\times\mathbb{C}$:

$$(\nabla_\zeta\nu)(m_o, (X_{q^i})) = \zeta_{m_o}\nu(m, (X_{q^i}\big|_m)) = \zeta_{m_o}\dot{\nu}(m)$$

$$\Longleftrightarrow (\nabla_\zeta\nu)^{\bullet} = \zeta\dot{\nu}\ .$$

<u>1C. Quantization</u>: the quantum bundle QB is defined by

$$QB = L\otimes B^D \cong (M\times\mathbb{C})\otimes(M\times\mathbb{C}) \cong M\times\mathbb{C}$$

and the global non-vanishing sections s_o and ν_o define the trivializing section $\psi_o = s_o \otimes \nu_o$ of QB :

$$\psi = s \otimes \nu = (\dot{s} s_o) \times (\dot{\nu}\nu_o) = (\dot{s}\dot{\nu})\psi_o = \dot{\psi}\psi_o .$$

With these definitions the partial connection on QB becomes

$$\nabla_\zeta s \otimes \nu = (\nabla_\zeta s) \otimes \nu + s \otimes \nabla_\zeta \nu = ((\zeta\dot{s} - \frac{i}{\hbar}\theta(\zeta)\dot{s})\dot{\nu} + \dot{s}\zeta\dot{\nu})\psi_o$$

hence $\quad (\nabla_\zeta \psi)^\cdot = \zeta\dot{\psi} - \frac{i}{\hbar}\theta(\zeta)\dot{\psi}$

Since ∇ is defined only for $\zeta \in D$ and since $\theta(D) = 0$ if follows:

$$(\nabla_\zeta \psi)^\cdot = \zeta\dot{\psi} ;$$

consequently sections of QB which are covariant constant along D can be identified with functions on $M = \mathbb{R}^{2n}$ which are independent of the coordinates p_i. The density associated to two such sections is defined by:

$$\begin{aligned}(\psi_1,\psi_2)\left(q^i, \left(\frac{\partial}{\partial q^j}\right)\right) &= (s_1,s_2)(m)\,\overline{\nu_1(m, (X_{q^j}))}\,\nu_2(m, (X_{q^j})) \cdot \\ &\quad \cdot \left|\varepsilon_\omega\left(X_{q^j}, \frac{\partial}{\partial q^i}\right)\right| \\ &= \overline{\dot{s}_1(m)}\,\dot{s}_2(m)\,\overline{\dot{\nu}_1(m)}\,\dot{\nu}_2(m) \\ &= \overline{\dot{\psi}_1(m)}\,\dot{\psi}_2(m)\end{aligned}$$

which is independent of the choice of $m \in \pi^{-1}(q^i)$ because the ψ are independent of the coordinates p_i. It follows that

$$< \psi_1,\psi_2 > = \int_{\mathbb{R}^n} \overline{\dot{\psi}_1(q)} \cdot \dot{\psi}_2(q)\,d\lambda^{(n)}$$

hence the Hilbert space determined by this quantization procedure is given by

$$\mathcal{H} = L^2(\mathbb{R}^n, \lambda^{(n)}) = \text{square integrable functions of the coordinates } (q^i).$$

The condition on an observable f to be quantizable is given by $[X_f, D^V] \subset D^V$ or equivalently

$$\frac{\partial^2 f}{\partial p_i \partial p_j} = 0 \qquad \forall i,j = 1,\ldots,n$$

$$\Longleftrightarrow \quad f(q,p) = f^o(q) + p_i f^i(q).$$

In order to define $\delta(f)$ for all quantizable observables we have to calculate the $-\frac{1}{2}$-D-density $L_\zeta \nu_o$ for vector fields ζ on M satisfying $[\zeta, D^V] \subset D^V$. Let ρ_t be the flow associated to ζ then:

$$\rho_{t*}\frac{\partial}{\partial p_i} = \frac{\partial(p_k \circ \rho_t)}{\partial p_i}\frac{\partial}{\partial p_k} = (\rho_t)_{ki}\frac{\partial}{\partial p_k}\ ;$$

there is no part $\frac{\partial}{\partial q^k}$ because $[\zeta, D^V] \subset D^V$ and:

$$\zeta = \zeta^i\frac{\partial}{\partial q^i} + \zeta_j\frac{\partial}{\partial p_j} \ \Rightarrow\ \frac{d}{dt}\Big|_{t=0}(\rho_t)_{ki} = \frac{\partial \zeta_k}{\partial p_i}.$$

Now we can compute $L_\zeta \nu_o$:

$$(L_\zeta \nu_o)\left(m_o, \left(-\frac{\partial}{\partial p_i}\right)\right) = \frac{d}{dt}\Big|_{t=0}\nu_o\left(\rho_t m_o, \rho_{t*}\left(-\frac{\partial}{\partial p_i}\right)\right)$$

$$= \frac{d}{dt}\Big|_{t=0}\nu_o\left(\rho_t m_o, \left(-\frac{\partial}{\partial p_k}\right)(\rho_t)_{ki}\right) = \frac{d}{dt}\Big|_{t=0}|\det \rho_t|^{-\frac{1}{2}}$$

$$= -\tfrac{1}{2}\left[|\det \rho_t|^{-\frac{1}{2}}\cdot(\det \rho_t)^{-1}\right]\Big|_{t=0}\cdot\frac{d}{dt}\det \rho_t$$

$$= -\tfrac{1}{2}\,\frac{\partial \det \rho_t}{\partial(\rho_t)_{ki}}\Big|_{t=0}\frac{d}{dt}\Big|_{t=0}(\rho_t)_{ki} = -\tfrac{1}{2}\,\delta_{ki}\frac{\partial \zeta_k}{\partial p_i}$$

$$\Longleftrightarrow (L_\zeta \nu_o)^{\cdot} = -\tfrac{1}{2}\sum_{i=1}^{n}\frac{\partial \zeta_i}{\partial p_i}.$$

For quantizable observables $f = f^o + p_i f^i$ we have

$$X_f = f^i(q)\frac{\partial}{\partial q^i} - \left(\frac{\partial f_o}{\partial q^i} + p_j\frac{\partial f^j}{\partial q^i}\right)\frac{\partial}{\partial p_i}$$

hence

$$\delta(f)\psi = \delta(f)\dot{\psi}\, s_o \otimes \nu_o$$
$$= (-i\hbar \nabla_{X_f} \dot{\psi}\, s_o + f\dot{\psi}\, s_o) \otimes \nu_o + \dot{\psi}\, s_o \otimes (-i\hbar\, L_{X_f} \nu_o)$$
$$= \left(-i\hbar X_f\dot{\psi} + f^o\dot{\psi} - \tfrac{1}{2} i\hbar \left(\sum_{k=1}^{n} \frac{\partial f^k}{\partial q^k}\right)\dot{\psi}\right) s_o \otimes \nu_o$$

so we can represent $\delta(f)$ on $L^2(\mathbb{R}^n, \lambda^{(n)})$ as

$$\delta(f^o(q)+p_i f^i(q))\dot{\psi}(q) = f^o(q)\dot{\psi}(q) - i\hbar\left(X_f\dot{\psi} + \tfrac{1}{2}\sum_k \frac{\partial f^k}{\partial q^k}\dot{\psi}\right)$$
$$= f^o(q)\dot{\psi}(q) - \tfrac{1}{2} i\hbar \left(\frac{\partial}{\partial q^k}(f^k(q)\dot{\psi}(q)) + f^k(q)\frac{\partial\dot{\psi}(q)}{\partial q^k}\right)$$

, because $\frac{\partial\dot{\psi}}{\partial p_j} = 0$ and if $\dot{\psi}$ is sufficiently differentiable.

<u>1D. Summary and conclusions</u>: $M = \mathbb{R}^{2n}$, $L = M \times \mathbb{C}$, $D = D^v$ imply in this quantization procedure

$$H = L^2(\mathbb{R}^n, \lambda^{(n)}) = \text{functions of } (q^i),$$

$f: M \to \mathbb{R}$ is quantizable iff $f(q,p) = f^o(q) + p_i f^i(q)$ and then we have for $\dot{\psi} \in H$:

$$\delta(f)\dot{\psi} = f^o(q)\dot{\psi}(q) + \tfrac{1}{2}\left(-i\hbar\frac{\partial}{\partial q^k}(f^k(q)\dot{\psi}(q)) + f^k(q)\left(-i\hbar\frac{\partial}{\partial q^k}\right)\dot{\psi}(q)\right)$$

or
$$\delta(f^o+p_k f^k) = f^o + \tfrac{1}{2}\left(-i\hbar\frac{\partial}{\partial q^k} f^k + f^k \cdot \left(-i\hbar\frac{\partial}{\partial q^k}\right)\right)$$

and especially $\delta(q^j) = q^j$, $\delta(p_j) = -i\hbar\frac{\partial}{\partial q^j}$

We see that the quantization procedure leads us, in this example, to the familiar Schrödinger quantization in the position representation: $H = L^2(\mathbb{R}^n)$, the position q^i is represented as multiplication by q^j and momentum p_j is represented as differentiation with respect to q^j (times $-i\hbar$).

Unfortunately, the class of quantizable observables is quite small: it does not contain the usual kinetic energy function $K = \Sigma p_i^2$. On the other hand, it does contain products of position and momentum and then the quantization prescription says: symmetrize the expression (in p and q) and substitute $-i\hbar\partial_q$ for p, which is the most natural way to do!

EXAMPLE 2: *Schrödinger quantization: momentum representation.*

The only (major) difference between this example and example 1 is that we choose $D = D^h$, $D^h_m = \left\{\frac{\partial}{\partial q^i}\right\}$ instead of $D = D^v$. The calculations are completely similar to those of example 1 so we merely list the results.

2A,B. Prequantization and the bundle B^D: $L = M \times \mathbb{C}$

$$s_o(m) = (m, 1) \; ; \quad s(m) = (m, \dot{s}(m)) \Longleftrightarrow s = \dot{s} \circ s_o$$

$$(\nabla_\zeta s)^\cdot = \zeta\dot{s} - \frac{i}{\hbar}\theta'(\zeta)\dot{s} \quad \text{with} \quad \theta' = -q^i dp_i .$$

N.B. We have chosen a different symplectic potential, which reflects a gauge transformation with gauge-factor $\exp(i\, p_k q^k/\hbar)$.

$$\nu_o\left(m, \left(\frac{\partial}{\partial q^i}\right)\right) = \nu_o(m, (X_{p_i})) = 1$$

$$\nu = \dot{\nu} \circ \nu_o \Rightarrow (\nabla_\zeta \nu)^\cdot = \zeta\dot{\nu} .$$

2D. Summary and conclusions: $M = \mathbb{R}^{2n}$, $L = M \times \mathbb{C}$, $D = D^h$ imply

$$H = L^2(\mathbb{R}^n, \lambda^{(n)}) = \text{square-integrable functions of the coordinates } (p_i),$$

$f: M \to \mathbb{R}$ is quantizable iff $f(q,p) = f_o(p) + q^k f_k(p)$ and for $\dot{\psi} \in H$

$$\delta(f)\dot{\psi} = f_o(p)\dot{\psi}(p) + \tfrac{1}{2}\left(i\hbar\frac{\partial}{\partial p_k}(f_k(p)\dot{\psi}(p)) + f_k(p)\left(i\hbar\frac{\partial}{\partial p_k}\right)\dot{\psi}(p)\right)$$

or $$\delta(f_o + q^k f_k) = f_o + \tfrac{1}{2}\left(i\hbar\frac{\partial}{\partial p_k} f_k + f_k\, i\hbar\frac{\partial}{\partial p_k}\right)$$

and especially $\delta(q^j) = i\hbar\frac{\partial}{\partial p_j}$, $\delta(p_j) = p_j$.

In this example we recover the Schrödinger quantization in the momentum representation; the class of quantizable observables differs from the previous case: arbitrary functions of p can be quantized and those which are linear in q. Now the kinetic energy can be quantized, but not a reasonable potential well !

It is well-known that there exists a relation between the two representations: the Fourier transform. One might suspect that in general a relation between the Hilbert spaces from two different polarizations is given as a unitary transformation between them. In geometric quantization this correspondence (if it exists!) is expressed by means of the so-called Blattner-Kostant-Sternberg-kernel (BKS-kernel), which is not treated in these notes.

<u>EXAMPLE 3</u>: *Schrödinger quantization of an arbitrary configuration space* $Q : M = T^*Q$.

In this example we quantize the symplectic manifold $M = T^*Q$, $\omega = d\theta$ (θ the canonical 1-form on T^*Q) by means of the vertical polarization D^V. If we have coordinate charts U_i with coordinates (q^j) on Q then they define charts $U_i \times \mathbb{R}^n$ of T^*Q $(\dim Q = n)$ with coordinates (q^i, p_j) for which $\theta = p_j dq^j$. Since $T^*U_i \cong U_i \times \mathbb{R}^n$ the quantization is locally the same as in example 1; minor differences occur in the global configuration.

<u>3A. Prequantization</u>: $L = M \times \mathbb{C} = T^*Q \times \mathbb{C}$ with a global non-vanishing section s_o :

$$s_o(m) = (m, 1)$$

and (as before) each section s determines a function $\dot{s}$:

$$s(m) = (m, \dot{s}(m)) \iff s = \dot{s} \cdot s_o .$$

The connection ∇ on L is defined by

$$(\nabla_\zeta s)^{\cdot} = \zeta\dot{s} - \frac{i}{\hbar}\,\theta(\zeta)\dot{s}$$

and the inner product: $(s_1,s_2)(m) = \overline{\dot{s}_1(m)} \cdot \dot{s}_2(m)$.

3B. The real polarization and the bundle B^D: we choose the vertical polarization $D = D^v$ defined on the local charts $U_i \times \mathbb{R}^n$ by

$$D^v_m = \left\{ \frac{\partial}{\partial p_i}\Big|_m \right\}$$

hence $M/D^v \cong Q$. For an arbitrary configuration space no obvious nowhere vanishing section of B^D exists (although B^D is trivial), so we use local trivializations of B^D by means of the local sections ν_i on $U_i \times \mathbb{R}^n$ defined by

$$\nu_i\left(m, \left(-\frac{\partial}{\partial p_j}\right)\Big|_m\right) = 1 .$$

With these local sections each $-\frac{1}{2}$-D-density ν on M defines a set of complex functions $\dot{\nu}^i$ on $U_i \times \mathbb{R}^n$:

$$\dot{\nu}^i(m) = \nu\left(m, \left(-\frac{\partial}{\partial p_j}\Big|_m\right)\right) \iff \nu\Big|_{U_i\times\mathbb{R}^n} = \dot{\nu}^i \cdot \nu_i \quad \text{(no summation!)}$$

In terms of these local trivializations we have:

$$(\nabla_\zeta \nu)^{\cdot i} = \zeta\dot{\nu}^i .$$

3C. Quantization: $QB = L \otimes B^D$ which has local trivializing sections $s_o \otimes \dot{\nu}^i$; for an arbitrary section ψ of QB we have:

$$\psi\Big|_{U_i\times\mathbb{R}^n} = \dot{s}s_o \otimes \dot{\nu}^i\nu_i = \dot{s}\dot{\nu}^i\, s_o \otimes \nu_i = \dot{\psi}^i\, s_o \otimes \nu_i .$$

The partial connection on QB is defined by

$$\nabla_\zeta(\dot{\psi}^i s_o \otimes \nu_i) = (\zeta\dot{\psi}^i - \frac{i}{\hbar}\theta(\zeta)\dot{\psi}^i)\, s_o \otimes \nu_i$$

but since $\zeta \in D$ and $\theta(D) = 0$ it follows:

$$(\nabla_\zeta \psi)^{\cdot i} = \zeta\dot{\psi}^i .$$

We see that a section ψ covariant constant along D can be identified with a set of functions $\dot{\psi}^i$ on U_i (since $U_i \times \mathbb{R}^n/D^v \cong U_i$); we will show that such a set of functions defines a (unique) ½-density on Q by using the relation between the $\dot{\psi}^i$'s for different indices i. If ψ is a section of QB covariant constant along D then we define the ½-density ϕ on Q locally on U_i (with coordinates (q^i)) by

$$\phi(x, \left(\frac{\partial}{\partial q^j}\right)) = \dot{\psi}^i(x) \quad , \qquad x \in U_i .$$

On the intersection $(U_i \times \mathbb{R}^n) \cap (U_j \times \mathbb{R}^n) \subset M = T^*Q$ there exist two coordinate systems: (q^j, p_j) from $U_i \times \mathbb{R}^n$ and $(\hat{q}^j, \hat{p}_j)$ from $U_j \times \mathbb{R}^n$ which are related by the formula

$$\hat{q}^j = \hat{q}^j(q^k) \qquad , \qquad p_j = \frac{\partial \hat{q}^k}{\partial q^j}\hat{p}_k$$

$$\Rightarrow \quad \frac{\partial}{\partial q^j} = \frac{\partial \hat{q}^k}{\partial q^j}\frac{\partial}{\partial \hat{q}^k} \qquad , \qquad \frac{\partial}{\partial \hat{p}_j} = \frac{\partial \hat{q}^j}{\partial q^k}\frac{\partial}{\partial p_k}$$

hence for $x \in U_i \cap U_j$, $m \in \pi^{-1}(x)$ we have

$$\begin{aligned}\phi(x, \left(\frac{\partial}{\partial \hat{q}^j}\right)) &= \dot{\psi}^j(x) = \psi(m, \left(-\frac{\partial}{\partial \hat{p}_j}\Big|_m\right)) = \psi(m, \left(-\frac{\partial}{\partial p_k}\Big|_m\right)\frac{\partial \hat{q}^j}{\partial q^k}) \\ &= \psi(m, \left(-\frac{\partial}{\partial p_k}\Big|_m\right)) \cdot \left|\det\frac{\partial \hat{q}}{\partial q}\right|^{-\frac{1}{2}} = \dot{\psi}^i(x) \cdot \left|\det\frac{\partial q}{\partial \hat{q}}\right|^{\frac{1}{2}} \\ &= \phi(x, \left(\frac{\partial}{\partial q^i}\right)) \cdot \left|\det\frac{\partial q}{\partial \hat{q}}\right|^{\frac{1}{2}}\end{aligned}$$

so ϕ is correctly defined as a $\frac{1}{2}$-density on Q.

Let ψ_1 and ψ_2 be two sections of QB covariant constant along D and let ϕ_1 and ϕ_2 be the associated $\frac{1}{2}$-densities on Q, then the density (ψ_1,ψ_2) on Q equals $\bar{\phi}_1\phi_2$; on U_i :

$$(\psi_1,\psi_2)(x, \left(\frac{\partial}{\partial q^j}\right)) = (\dot{\psi}_1^i s_o, \dot{\psi}_2^i s_o)(m) \cdot \overline{\nu_i(m, \left(-\frac{\partial}{\partial p_j}\right))} \cdot$$

$$\cdot \nu_i(m, \left(-\frac{\partial}{\partial p_j}\right)) \cdot \left|\varepsilon_\omega\left(-\frac{\partial}{\partial p_j}, \frac{\partial}{\partial q^j}\right)\right|$$

$$\Longleftrightarrow (\psi_1,\psi_2)(x, \left(\frac{\partial}{\partial q^j}\right)) = \overline{\dot{\psi}_1^i(x)}\, \dot{\psi}_2^i(x) = (\bar{\phi}_1\phi_2)\,(x, \left(\frac{\partial}{\partial q^j}\right)) .$$

We conclude that the identification of sections ψ of QB covariant constant along D with $\frac{1}{2}$-densities ϕ on Q carries over to the inner product, hence we can identify the Hilbert space of this quantization procedure with the square integrable $\frac{1}{2}$-densities on Q :

$$H = \{\phi: Q \to \Delta^{\frac{1}{2}}Q \mid \int_Q |\phi|^2 < \infty\} .$$

To identify the quantizable observables, we start with local considerations: on $T^*U_i \cong U_i \times \mathbb{R}^n$ the condition $[X_f, D^V] \subset D^V$ is equivalent with

$$f(q,p) = f^o(q) + p_i f^i(q)$$

(see example 1) which is the sum of a function on U_i and a function linear over $\mathbb{R}^n$ (from $U_i \times \mathbb{R}^n$). If we concentrate our attention to the part linear over $\mathbb{R}^n$, we note that the property "linear over $\mathbb{R}^n$ " is conserved by the transition functions from $U_i \times \mathbb{R}^n$ to $U_j \times \mathbb{R}^n$ and (moreover) such a "linear function" defines a vector field ξ on Q :

$$\xi\big|_{U_i} = f^i(q)\frac{\partial}{\partial q^i} .$$

On the other hand, each vector field ξ on Q defines a function f_ξ on T^*Q which is linear over $\mathbb{R}^n$ (on each $U_i \times \mathbb{R}^n$) as follows:

$$\alpha \in T^*Q \;\Rightarrow\; f_\xi(\alpha) = \alpha(\xi|_{\pi(\alpha)})$$

or, in local coordinates (q,p) on $U_i \times \mathbb{R}^n$:

$$\xi = \xi^i(q)\frac{\partial}{\partial q^i}\ , \qquad \alpha = p_i dq^i|_{(q)} \;\Rightarrow\; f_\xi(\alpha) = p_i \xi^i(q).$$

As a consequence, each quantizable observable f on $M = T^*Q$ is of the form

$$f = f^o + f_\xi$$

where ξ is a real vector field on Q and f^o a function on Q (or more precisely: f^o is a function of the form $f^o = g \circ \pi$ where $g\colon Q \to \mathbb{R}$ and $\pi\colon T^*Q \to Q$ the natural projection).

Now suppose ξ is a vector field on Q, then it defines an operation on $\frac{1}{2}$-densities denoted by L_ξ (abuse of notation); its definition is completely analogous to the definition of L_ζ on $-\frac{1}{2}$-D-densities on M (see section 6): in a certain sense L_ξ is the natural action of ξ on $\frac{1}{2}$-densities.

Let ρ_t be the flow associated to ξ and let ϕ be a $\frac{1}{2}$-density on Q then:

$$(L_\xi\phi)(x, (\eta_j)) = \frac{d}{dt}\Big|_{t=0} \phi(\rho_t x, (\rho_{t*}\eta_j))$$

which is again a $\frac{1}{2}$-density on Q. In local coordinates (q^j) on $U_i \subset Q$ we can compute L_ξ as follows:

$$\xi = \xi^j(q)\frac{\partial}{\partial q^j}, \quad (\rho_t)^k_j := \frac{\partial q^k \circ \rho_t}{\partial q^j}\ , \quad \dot{\phi}^i(x) := \phi\left(x, \left(\frac{\partial}{\partial q^j}\Big|_m\right)\right)$$

and then $L_\xi\phi$ becomes:

$$\begin{aligned}(L_\xi\phi)\left(x,\left(\frac{\partial}{\partial q^j}\right)\right) &= \frac{d}{dt}\Big|_{t=0}\ \phi\left(\rho_t x,\left(\rho_{t*}\frac{\partial}{\partial q^j}\right)\right)\\
&= \frac{d}{dt}\Big|_{t=0}\ \phi\left(\rho_t x,\left(\frac{\partial}{\partial q^k}\Big|_{\rho_t x}\right)(\rho_t)^k_j\right)\\
&= \frac{d}{dt}\Big|_{t=0}\ \dot\phi^i(\rho_t x)\cdot\left|\det(\rho_t)^k_j\right|^{\frac12}\\
&= \xi\,\dot\phi^i + \dot\phi^i(x)\,\frac{d}{dt}\Big|_{t=0}\left|\det(\rho_t)^k_j\right|^{\frac12}\\
&= (\xi\,\dot\phi^i)(x) + \dot\phi^i(x)\cdot\tfrac12\sum_{k=1}^n \frac{\partial\xi^k}{\partial q^k}(x)\ .\end{aligned}$$

If ψ is a section of QB covariant constant along D and ϕ the associated $\frac12$-density on Q then we know (from example 1):

$$\begin{aligned}(\delta(f_\xi)\psi)^{\cdot i} &= -\tfrac12 i\hbar\left(\frac{\partial}{\partial q^k}(\xi^k(q)\dot\psi^i(q)) + \xi^k(q)\,\frac{\partial\dot\psi^i}{\partial q^k}(q)\right)\\
&= -i\hbar\left(\xi\,\dot\psi^i + \tfrac12\dot\psi^i\sum_{k=1}^n\frac{\partial\xi^k}{\partial q^k}(q)\right)\end{aligned}$$

and we know also (from above):

$$(L_\xi\phi)^{\cdot i} = \xi\,\dot\phi^i + \tfrac12\dot\phi^i\sum_{k=1}\frac{\partial\xi^k}{\partial q^k}\ ,$$

so if we remember that $\dot\phi^i(x) = \phi\left(x,\left(\frac{\partial}{\partial q^j}\right)\right) = \dot\psi^i(x)$ then

$$(\delta(f_\xi)\psi)^{\cdot i} = -i\hbar(L_\xi\phi)^{\cdot i}$$

or equivalently: the section $\delta(f_\xi)\psi$ corresponds to the $\frac12$-density $-i\hbar L_\xi\phi$.

<u>3D. Summary</u>: $M = T^*Q$, $L = M\times\mathbb{C}$, $D = D^V$ imply:

$$H = \{\text{square integrable } \tfrac12\text{-densities on } Q\}$$

$f\colon M\to\mathbb{R}$ is quantizable iff $f = f^o + f_\xi$ where f^o is a function "on Q" and ξ a vector field on Q ; for $\phi\in H$ we have

$$\delta(f^o+f_\xi)\phi = f^o\phi - i\hbar L_\xi\phi$$

or $$\delta(f^o+f_\xi) = f^o - i\hbar L_\xi \ .$$

EXAMPLE 4: *Schrödinger quantization of* S^2.

This example is a special case of example 3 in which we will give another interpretation of the Hilbert space and the operators associated to quantizable observables. Although the forthcoming interpretation can be made for any Riemannian manifold Q, we specialize to S^2 because it is a low-dimensional orientable manifold which enables us to give "actual computations".

4C. Quantization: on S^2 there exists a natural volume element ε associated to the metric on S^2; in local coordinates (θ,ϕ) defined by:

$$\mathbb{R}^3 \supset S^2 \ni \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \sin\theta \ \sin\phi \\ \sin\theta \ \cos\phi \\ \cos\theta \end{pmatrix}$$

it is given by

$$\varepsilon = \sin\theta \ d\theta \wedge d\phi.$$

With the aid of this volume element we define a global trivializing section ϕ_o of $\Delta^{\frac{1}{2}} S^2$ as follows: for any local coordinate system (q^1,q^2) on S^2:

$$\phi_o(x, \left(\frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}\right)) := \left|\varepsilon_x\left(\frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}\right)\right|^{\frac{1}{2}}$$

and it should be obvious that ϕ_o is a nowhere vanishing $\frac{1}{2}$-density on S^2 (because ε is a nowhere vanishing 2-form). With the aid of ϕ_o we can identify each $\frac{1}{2}$-density ϕ on S^2 with a function $\dot{\phi}$ on S^2 by $\phi = \dot{\phi}\phi_o$ and we have:

$$\int_{S^2} \bar{\phi}_1 \phi_2 = \int_{S^2} \bar{\dot{\phi}}_1 \dot{\phi}_2 \varepsilon$$

where the second integration should be done over oriented charts (with respect to ε). This identification enables us to interpret the Hilbert space as

$$H = \{\dot{\phi}: S^2 \to \mathbb{C} \mid \int_{S^2} |\phi|^2 \varepsilon < \infty\}$$

i.e. as square integrable functions on S^2.

To compute the action of a quantizable observable we make the following remark: if ξ is a vector field on S^2 then $L_\xi \phi_o$ is a ½-density on S^2 which depends only on ξ (and ϕ_o of course), hence it defines a function $\mathrm{div}(\xi)$ on S^2 by:

$$L_\xi \phi_o = \tfrac{1}{2} \cdot \mathrm{div}(\xi) \phi_o ;$$

$\mathrm{div}(\xi)$ is called the divergence of the vector field ξ, the extra factor ½ comes in because ϕ_o is a ½-density. The divergence of a vector field can be defined on any Riemannian manifold; on $\mathbb{R}^n$ it yields the usual value of the divergence of a vector field when expressed in carthesian coordinates, but the formula above gives an intrinsic definition. To be more specific, if u is the natural volume form on an oriented Riemannian manifold, then on any coordinate chart $(x^1, \dots, x^n)$ it defines a function Vol by

$$\mathrm{Vol}(x^1, \dots, x^n) = \left| u_x\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) \right| ;$$

if $\eta = \eta^i(x) \frac{\partial}{\partial x^i}$ is a vector field then:

$$\mathrm{div}(\eta) = \sum_{i=1}^{n} \frac{\partial \eta^i}{\partial x^i}(x) + \frac{1}{\mathrm{Vol}} \eta^i \frac{\partial \,\mathrm{Vol}}{\partial x^i} .$$

On S^2 with coordinates (θ,ϕ) as defined above the divergence of a vector field ξ is defined by:

$$\xi = \xi^\theta \frac{\partial}{\partial\theta} + \xi^\phi \frac{\partial}{\partial\phi} \;\Rightarrow\; \mathrm{div}(\xi) = \frac{\partial\xi^\theta}{\partial\theta} + \frac{\partial\xi^\phi}{\partial\phi} + \xi^\theta \operatorname{cotg}\theta \,.$$

Now if $\phi = \dot\phi\phi_o$ is any $\frac{1}{2}$-density and ξ any vector field on S^2 then

$$L_\xi\phi = (\xi\dot\phi + \tfrac{1}{2}\dot\phi\,\mathrm{div}(\xi))\phi_o$$

hence the action of a quantizable observable $f^o + f_\xi$ on the Hilbert space of square integrable functions on S^2 is given by

$$\delta(f^o+f_\xi)\dot\phi = (f^o - \tfrac{1}{2}i\hbar\,\mathrm{div}(\xi))\dot\phi - i\hbar\xi\dot\phi$$

<u>4D. Summary and conclusions:</u> $M = T^*S^2$, $L = M\times\mathbb{C}$, $D = D^v$ imply

$$H = L^2(S^2,\varepsilon) = \text{square integrable functions on } S^2 \text{ with respect to the volume } \varepsilon\,,$$

$f\colon T^*S^2 \to \mathbb{R}$ is quantizable iff $f = f^o + f_\xi$ and then for $\dot\phi \in H$:

$$\delta(f^o+f_\xi) = f^o - \tfrac{1}{2}\hbar\,\mathrm{div}(\xi) - i\hbar\xi$$

If we apply this method to the Riemannian manifold $Q = \mathbb{R}^n$, we get three "different" interpretations of the Schrödinger quantization in the position representation: in example 1 we had $H = L^2(\mathbb{R}^n)$ and the quantizable observables were quantized in the old fashioned way using the symmetrized form if a momentum coordinate was involved (only linear!); this quantization is the most intuitive one, although it depends strongly on the carthesian coordinates. After applying the techniques of example 4 we get the same quantization, although now the action of quantizable observables is defined in an intrinsic way, independent of a coordinate system.

In example 3 we had $\mathcal{H}$ = {square integrable ½-densities} with the action of $\delta(f_\xi)$ the natural action of ξ on ½-densities; in a certain sense, this quantization is the most elegant of the three, but also the least intuitive one.

EXAMPLE 5: *the 1-dimensional harmonic oscillator in the energy representation.*

In this example we try to quantize the symplectic manifold (M,ω) where $M = \mathbb{R}^2 \setminus \{(0,0)\}$ with coordinates (q,p) and symplectic form $dp \wedge dq$. The reason for omitting the origin is that we want to use the circular polarization D^c defined by

$$D^c_{(q,p)} = \mathbb{R}\left(p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p}\right) = \mathbb{R}X_H ,$$

where $H = \frac{1}{2}(p^2+q^2)$ is the hamiltonian of a 1-dimensional (normalized) harmonic oscillator. Since $D^c_{(0,0)}$ is 0-dimensional, D^c does not define a real polarization on $\mathbb{R}^2$, hence we omit the origin and we hope it will not influence our results very much. In view of the circular polarization it is easier to use polar coordinates (ϕ,ρ) on $M = \mathbb{R}^2 \setminus \{(0,0)\}$ defined by

$$p = \sqrt{2\rho}\,\cos\phi \qquad\qquad \rho = H = \tfrac{1}{2}(p^2+q^2)$$

$$q = \sqrt{2\rho}\,\sin\phi \qquad\qquad \phi = \operatorname{arctg}(q/p)$$

then $D^c = \mathbb{R}\left(\frac{\partial}{\partial\phi}\right)$ and $M/D^c \simeq \mathbb{R}^+ = (0,\infty)$ with projection $\pi : (\phi,\rho) \longmapsto \rho$. It should be noted that ϕ is *not* a global coordinate, hence we should have used two coordinate patches $\phi \in (0,2\pi)$ and $\phi \in (\pi,3\pi)$; however, we will use $\phi \in [0,2\pi]$ and impose the conditions that $\phi = 0$ and $\phi = 2\pi$ determine the same results; moreover, the vector field $\frac{\partial}{\partial\phi}$ and the 1-form $d\phi$ are defined globally!

5A. Prequantization: $L = M \times \mathbb{C}$ with global non-vanishing section s_o : $s_o(m) = (m, 1)$ which identifies sections with functions by

$$s(m) = (m, \dot{s}(m)), \qquad s = \dot{s} \cdot s_o .$$

The connection is given by

$$(\nabla_\xi s)^{\cdot} = \zeta\dot{s} - \frac{i}{\hbar}\theta(\zeta)\dot{s} , \qquad \theta = \rho d\phi$$

(N.B. θ is globally defined and $d\theta = d\rho \wedge d\phi = dp \wedge dq$), and the inner product by

$$(s_1,s_2)(m) = \overline{\dot{s}_1(m)}\, \dot{s}_2(m) .$$

With these ingredients, the prequantization Hilbert space can be identified with $L^2(\mathbb{R}^2, \lambda^{(2)})$.

5B. The bundle B^{D^c}: the bundle B^{D^c} has an obvious non-vanishing section ν_o defined by

$$\nu_o(m,X_\rho) = 1$$

which identifies B^D with $M \times \mathbb{C}$ by:

$$\nu(m,X_\rho) = \dot{\nu}(m), \qquad \nu = \dot{\nu} \cdot \nu_o$$

and we have (as before)

$$(\nabla_\zeta \nu)^{\cdot} = \zeta\dot{\nu}$$

5C. Quantization? the section s_o of L and ν_o of B^{D^c} define a global non-vanishing section of $QB = L \otimes B^{D^c}$ by $\psi_o = s_o \otimes \nu_o$, hence if ψ is a section of QB it defines a unique function $\dot{\psi}$ on M (and vice-versa):

$$\psi = \dot{\psi} \cdot \psi_o ,$$

and the partial connection on QB is defined by

$$(\nabla_\zeta \psi)^\bullet = \zeta\dot{\psi} - \frac{i}{\hbar}\theta(\zeta)\dot{\psi}\,, \qquad \theta = \rho d\phi\,.$$

Furthermore, if ψ_1 and ψ_2 are two sections of QB covariant constant along D^c, then they define a density on $\mathbb{R}^+$:

$$(\psi_1,\psi_2)\left(\rho,\frac{\partial}{\partial\rho}\right) = \overline{\dot{\psi}_1(\rho,\phi)}\dot{\psi}_2(\rho,\phi)\ \overline{\nu_o\left(\frac{\partial}{\partial\phi}\right)}\nu_o\left(\frac{\partial}{\partial\phi}\right)\cdot\left|\varepsilon_\omega\left(\frac{\partial}{\partial\phi},\frac{\partial}{\partial\rho}\right)\right|$$

$$= \overline{\dot{\psi}_1(\rho,\phi)}\dot{\psi}_2(\rho,\phi)$$

hence $\quad <\psi_1,\psi_2> = \int_o^\infty \overline{\dot{\psi}_1(\rho,\phi)}\dot{\psi}_2(\rho,\phi)d\rho$

and it should be noted that $\overline{\dot{\psi}_1(\rho,\phi)}\dot{\psi}_2(\rho,\phi)$ is independent of ϕ because we supposed ψ_1 and ψ_2 covariant constant along D !

Before we investigate the condition "covariant constant along D", we first investigate the quantizable observables: f is quantizable iff $[X_f,D^c] \subset D^c$ which translates to

$$\frac{\partial^2 f}{\partial\phi^2} = 0 \iff f(\phi,\rho) = g(\rho) + \phi\cdot h(\rho)\,,$$

but now we realize that ϕ is a cyclic coordinate, hence $h(\rho) = 0$ so f is quantizable iff f depends only on ρ. But then

$$X_f = \frac{\partial f}{\partial\rho}\frac{\partial}{\partial\phi} - \frac{\partial f}{\partial\phi}\frac{\partial}{\partial\rho} = \frac{\partial f}{\partial\rho}\frac{\partial}{\partial\phi} \in D^c$$

which implies that

$$\delta(f)\psi = -i\hbar\nabla_{X_f}\psi + f\psi = f\psi$$

because if $\psi \in H$ then ψ is covariant constant along D.

The condition $\psi = \dot{\psi}\psi_o$ covariant constant along D translates as:

$$0 = (\nabla_\zeta \psi)^{\cdot} = \zeta\dot{\psi} - \frac{i}{\hbar}\theta(\zeta)\dot{\psi}\ , \qquad \theta = \rho d\phi, \qquad \zeta \in D = \left\{\frac{\partial}{\partial\phi}\right\}$$

$$\Longleftrightarrow\ \frac{\partial\dot{\psi}}{\partial\phi} = \frac{i}{\hbar}\rho\dot{\psi}\ \Longleftrightarrow\ \dot{\psi}(\rho,\phi) = h(\rho)\exp(i\rho\phi/\hbar).$$

But ... , ϕ is a cyclic coordinate hence $h(\rho) = h(\rho)\exp\left(\frac{2\pi i\rho}{\hbar}\right)$ which is equivalent with

$$h(\rho) = 0 \vee \frac{\rho}{\hbar} \in \mathbb{Z} \Longleftrightarrow \left[h(\rho) \neq 0 \Rightarrow \rho \in \hbar\mathbb{Z}^+\right]$$

where $\mathbb{Z}^+ = \{1,2,\ldots\}$ and $\rho = H$ is the hamiltonian function, so (if we forget for the moment that $\dot{\psi}$ should be continuous) a section ψ of QB co-variant constant along D is determined by a set complex numbers a_n

$$\psi \longleftrightarrow \{a_n\}_{n\in\mathbb{Z}^+}$$

where a_n is the value of $h(\rho)$ at $\rho = n\hbar$.
The action of the hamilton operator $H = \rho$ is (as we have seen) just a multiplication operator:

$$\delta(H)\{a_n\}_{n\in\mathbb{Z}^+} = \{b_n\}_{n\in\mathbb{Z}^+}\ , \quad b_n = n\hbar\, a_n$$

so the eigenvalue equation $\delta(H)\psi = E\psi$ has solutions ψ_n with eigenvalues E_n defined by

$$\psi_n \longleftrightarrow \{\delta^n_k\}_{k\in\mathbb{Z}^+}\ , \qquad E_n = n\hbar\ , \qquad n\hbar \in \mathbb{Z}^+\ ,$$

and the eigenfunctions form a basis in the sense:

$$\psi \longleftrightarrow \{a_n\}_{n\in\mathbb{Z}^+} \Longleftrightarrow \psi = \sum_{n=1}^{\infty} a_n\psi_n\ .$$

5D. Conclusions: this quantization procedure of the harmonic oscillator corresponds to the classical Bohr-Sommerfeld quantization of the harmonic oscillator which differs from the usual quantum mechanical answer by a term $\frac{1}{2}\hbar$

(E_n should be $(n-\frac{1}{2})\hbar$, $n \in \mathbb{Z}^+$). Woodhouse claims that the origin of this error (and of similar ones) can be traced back to the use of $-\frac{1}{2}$-D-densities instead of the so-called $-\frac{1}{2}$-D-forms; these will be treated later in these notes and we will show that their use leads us to the correct answer $E_n = (n-\frac{1}{2})\hbar$, $n \in \mathbb{Z}^+$.

However, although the above quantization seems rather nice (apart from the energy shift of $\frac{1}{2}\hbar$), it has one obvious drawback:

$$H = \{0\}$$

because: if ψ_1 and ψ_2 are two sections of QB covariant constant along D, then the density (ψ_1,ψ_2) is zero outside the set $\mathbb{Z}^+ \subset \mathbb{R}^+$, but $\mathbb{Z}^+$ has a zero-measure, hence $<\psi_1,\psi_2> = 0$. The solution to this problem would be to allow "distribution valued" sections of QB in such a way that the inner product comes out as a summation:

$$<\left\{a_n\right\}_{n\in\mathbb{Z}^+}, \left\{b_n\right\}_{n\in\mathbb{Z}^+}> = \sum_{n=1}^{\infty} \overline{a_n}\, b_n \, .$$

This approach is given (although brief) in [Woodhouse] and [Sniatycki] (see also [Guillemin & Sternberg Ch.VI]); it will not be a topic in these notes.

8 COMPLEX POLARIZATIONS

In the previous sections we introduced a real polarization in order to reduce "the size" of the quantum Hilbert space. In section 7 we saw that some modifications are needed to obtain the correct energy levels, but in order to describe these modifications we have to use the complexified tangent bundle everywhere instead of the usual (real) tangent bundle. Together with the use of complex vector fields one introduces the concept of a complex polarization, which is a (straightforward) generalization of a real polarization and it will give us more freedom in the choice of a polarization and hence in the "choice" of the set of quantizable observables.
This section will be devoted entirely to the study of complex polarizations and in this and all subsequent sections all vector field and k-forms will be complex, i.e. we will use consequently the bundles $TM^{\mathbb{C}}$ and $\Lambda^k M^{\mathbb{C}}$ instead of TM and $\Lambda^k M$; on the complexified bundles we will use the obvious complex conjugation denoted by $\overline{}$, e,g, if $v, w \in TM$ then $v+iw \in TM^{\mathbb{C}}$ and $\overline{v+iw} = v-iw$.

The most obvious definition of a complex polarization would be: a complex polarization is a complex distribution P which is integrable and lagrangian (= maximal isotropic). However, for complex distributions the notion of integrability is defined as follows: P is integrable if there exist (locally) independent complex functions $z_1, \ldots, z_k$ such that $P = \{v \mid \forall 1 \leq i \leq k : dz_i(v) = 0\}$, which is in general not equivalent to the statement that P is involutive (see [Nirenberg]). Hence at the moment it is not clear which characterization is to be used (integrable or involutive) so we generalize the other characterization of a real polarization: a distribution spanned (locally) by local hamiltonian vector fields whose associated functions "commute".

DEFINITION: a (complex) polarization P is a complex distribution on M such that for each $m_o \in M$ there is a neighbourhood U of m_o and n independent (C^∞) complex functions $z_1, \ldots, z_n$ on U such that:

(i) $\forall m \in U : P_m$ is spanned by $\{X_{z_1}, \ldots, X_{z_n}\}$ (over $\mathbb{C}$!)

(ii) on $U : [z_i, z_j] = 0$, $1 \leq i,j \leq n$ (the Poisson bracket extended to complex functions)

(iii) $\dim_{\mathbb{C}}(P \cap \bar{P}) = k$ constant on M

(iv) $\exists w_1, \ldots, w_k$ complex functions on $U : P \cap \bar{P}$ is spanned by $\{X_{w_1}, \ldots, X_{w_k}\}$ on U.

The condition (i) and (ii) are straightforward generalizations of the real case; conditions (iii) and (iv) are added to make life easier, i.e. to insure that we can do something. It should be noted that a real polarization D defines in an obvious way a complex polarization $P : P = D^{\mathbb{C}}$; the conditions (i) and (ii) are satisfied with real functions $z_1, \ldots, z_n$, $P = \bar{P}$ hence $k = n$ and we can take $w_i = z_i$. From this point of view it should be understandable that a complex polarization P with $\dim_{\mathbb{C}} P \cap \bar{P} = n$ is called real (because then there exists a real polarization D with $P = D^{\mathbb{C}}$); on the other hand, if $\dim_{\mathbb{C}} P \cap P = k = 0$ then the complex polarization P is called a Kähler polarization.

In the rest of this section we will study some properties of a complex polarization; we start with a sequence of propositions to derive an alternative description of a complex polarization.

PROPOSITION: *in condition* (iv) *of a complex polarization one can always assume that the* w_j *are real functions.*

PROOF: $X_{w_j} = X_{\mathrm{Re}\, w_j} + i X_{\mathrm{Im}\, w_j}$ hence $P \cap \bar{P}$ is spanned by $\{X_{\mathrm{Re}\, w_j}, X_{\mathrm{Im}\, w_j} \mid 1 \leq j \leq k\}$ of which k are independent (at least locally).

QED

DEFINITION: we define two real distributions D and E on M associated to P by:

$$D = P \cap \bar{P} \cap TM \quad (\text{hence } D^{\mathbb{C}} = P \cap \bar{P}, \ \dim_{\mathbb{R}} D = k)$$

$$E = (P+\bar{P}) \cap TM \quad (\text{hence } E^{\mathbb{C}} = P + \bar{P}, \ \dim_{\mathbb{R}} E = 2n - k)$$

PROPOSITION: *conditions* (i) *and* (ii) *of* P *imply that* P *is a maximal isotropic involutive complex distribution on* M.

COROLLARY: D *is involutive hence* D *is an integrable real distribution of dimension* k *on* M.

PROPOSITION: $E = \{v \in TM \mid \forall w \in D : \omega(v,w) = 0\}$

$D = \{w \in TM \mid \forall v \in E : \omega(v,w) = 0\}$.

PROOF: use that P is isotropic and proposition 5.3.2 of [Abraham & Marden].

PROPOSITION: $X_f \in D \iff$ f *constant along* E $(\iff \forall Y \in E : Yf = 0)$

$X_f \in E \iff$ f *constant along* D

COROLLARY: *condition* (iv) *of* P *is equivalent to the condition that* E *is an integrable (real) distribution.*

PROPOSITION: (see [Nirenberg]): *let* P *be a complex distribution of dimension* n *on* M *such that* $\dim_{\mathbb{C}} P \cap \bar{P}$ *is constant on* M *and such that both* P *and* $P + \bar{P}$ *are involutive, then there exist (locally) complex functions* $z_1, \ldots, z_n$ *on* M:

$$P = \{v \in TM^{\mathbb{C}} \mid dz_i(v) = 0, \ 1 \leqslant i \leqslant n\}$$

Using this proposition one can (easily) prove the following characterization of a complex polarization:

PROPOSITION: *a complex distribution* P *on* M *is a complex polarization if and only if:*

(i) P involutive

(ii) P is lagrangian ($\iff$ maximal isotropic)

(iii) $\dim_{\mathbb{C}} P \cap \bar{P} = k$ constant on M

(iv) $P + \bar{P}$ is involutive

REMARK: since $E^{\mathbb{C}} = P + \bar{P}$, the condition $P + \bar{P}$ involutive is equivalent to the condition E involutive which is in turn equivalent to the condition E integrable.

REMARK 2: some authors omit condition (iv) in their definition of a complex polarization (either in the form of our definition or in the form of the proposition above). However, this condition is sufficient to prove the equivalence of both descriptions of a complex polarization and, moreover, I need it to prove the correctness of the inproduct in the quantum Hilbert space (as we will see in the next section).

DEFINITION: a complex polarization P is called admissible if M/D admits the structure of a manifold such that $\pi: M \to M/D$ is a submersion ($\dim M/D = 2n - k$)

EXAMPLE 1: $M = T^*\mathbb{R} \cong \mathbb{R}^2$, $\omega = dp \wedge dq$, $P = \mathbb{C}\left(\frac{\partial}{\partial p} + i\frac{\partial}{\partial q}\right) \iff P = \mathbb{C} \cdot X_{p+iq}$; $P \cap \bar{P} = \{0\}$ so P is a Kähler polarization on M. Since $P \cap \bar{P} = \{0\}$ we have $D = \{0\}$, $E = \left\{\frac{\partial}{\partial p}, \frac{\partial}{\partial q}\right\} = TM$ and the projection $\pi: M \to M/D \cong M$ is obviously a submersion, so P is an admissible complex (Kähler) polarization.

EXAMPLE 2: $M = T^*\mathbb{R}^2 \cong \mathbb{R}^4$, $\omega = dp_1 \wedge dq^1 + dp_2 \wedge dq^2$, $P = \left\{\frac{\partial}{\partial p_1} + i\frac{\partial}{\partial q_1}, \frac{\partial}{\partial p_2}\right\} = \mathbb{C} \cdot X_{p_1+iq^1} \oplus \mathbb{C}\, X_{q^2}$; since $[p_1+iq^1, q^2] = 0$ this complex distribution satisfies conditions (i) and (ii) of a complex polarization. $P \cap \bar{P} = \mathbb{C} \cdot X_{q^2}$

so it also satisfies the conditions (iii) and (iv), hence P is a complex polarization. Now $D = \left\{X_{q^2}\right\} = \left\{\frac{\partial}{\partial p_2}\right\}$, $E = \left\{\frac{\partial}{\partial p_2}, \frac{\partial}{\partial p_1}, \frac{\partial}{\partial q^1}\right\}$ and $M/D \cong \mathbb{R}^3$ with $\pi(q^1, q^2, p_1, p_2) = (q^1, q^2, p_1)$ so P is an admissible complex polarization.

Nota Bene: in the sequel we will always use admissible complex polarizations on our symplectic manifold (M,ω), hence by "a polarization P" we always mean "an admissible complex polarization P", unless stated otherwise.

Our next aim is to prove some propositions concerning the relation between P, D, E and the structure of the manifold M/D; to be more specific, we want to show that for a polarization P:

(a) $\tilde{E} := \pi_* E$ is an integrable distribution of dimension $2(n-k)$ on M/D,

(b) P induces on each leaf L of $\tilde{E}$ the structure of a complex manifold of dimension $n-k$,

(c) complex functions z with $X_z \in P$ are precisely the functions $z = \tilde{z} \circ \pi$ where $\tilde{z}$ is a function on M/D which is holomorphic when restricted to a leaf of $\tilde{E}$ (seen as a complex manifold).

LEMMA (Frobenius enlarged): if D is an involutive real distribution of dimension ℓ on a manifold X $(\dim X = m)$, and if U is a coordinate neighbourhood of $x \in X$ with coordinates $x^1, \ldots, x^m$ such that there exists a k $(0 \leqslant k \leqslant \ell)$: $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^k} \in D$ on U, then there exists a coordinate neighbourhood V of x, $V \subset U$ with coordinates $y^1, \ldots, y^m$ such that D is spanned by $\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{y^\ell}$ and moreover $y^i = x^i$, $1 \leqslant i \leqslant k$ and $\frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}$, $1 \leqslant i \leqslant k$.

PROOF: the proof is a slight modification of the proof of the classical Frobenius theorem as can be found in [Godbillon]. Note that the case $k = 0$ is the classical Frobenius theorem. QED

PROPOSITION: *the set* $\tilde{E} = \pi_* E$ *is a well-defined integrable distribution of dimension* $2(n-k)$ *on* M/D.

PROOF: by applying the above lemma twice we are assured, for each $m_o \in M$, of the existence of a neighbourhood U of m_o with coordinates $x^1, \ldots, x^{2n}$ on U such that D is spanned by $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^k}\right\}$ and E is spanned by $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{2n-k}}\right\}$. Since π is a submersion we may assume that $\pi(U)$ is a neighbourhood of (m_o) with coordinates $(x^{k+1}, \ldots, x^{2n})$ and projection $\pi(x^1, \ldots, x^{2n}) = (x^{k+1}, \ldots, x^{2n})$. It follows that $\pi_* E$ is spanned by $\frac{\partial}{\partial x^{k+1}}, \ldots, \frac{\partial}{\partial x^{2n-k}}$ independent of the point m in the fibre of π, so $\tilde{E}$ defined by $\tilde{E} = \pi_* E$ is a well-defined distribution on M/D which is obviously integrable. QED

The next lemma will not be used in this section (it will in the next) but its proof uses the techniques of this section, which is the reason we put it down here, instead of in the next section.

LEMMA: suppose $X_{r_1}, \ldots, X_{r_k}, X_{z_{k+1}}, \ldots, X_{z_n}$ are hamiltonian vector fields on a neighbourhood U of $m_o \in M$ which span P on U such that $X_{r_1}, \ldots, X_{r_k}$ span D, then there exist vector fields $Y_1, \ldots, Y_k$ on a neighbourhood V $(m_o \in V \subset U)$ such that:

(i) $\pi_*(X_{z_{k+1}}, \ldots, X_{z_n}, X_{\bar{z}_{k+1}}, \ldots, X_{\bar{z}_n}, Y_1, \ldots, Y_k)$ is a basis of $T_{\pi(m)}(M/D)^{\mathbb{C}}$ which depends only on $\pi(m)$ $(\forall m \in V)$

(ii) $\varepsilon_\omega(X_{r_1}, \ldots, X_{r_k}, X_{z_{k+1}}, \ldots, X_{z_n}, X_{\bar{z}_{k+1}}, \ldots, X_{\bar{z}_n}, Y_1, \ldots, Y_k)$ and $\omega^{n-k}(X_{z_{k+1}}, \ldots, X_{z_n}, Z_{\bar{z}_{k+1}}, \ldots, X_{\bar{z}_n})$ are functions on V which are constant on the leaves of D.

PROOF: we use the coordinate system $(x^1, \ldots, x^{2n})$ of the proof of the previous proposition. The vectors $X_{r_1}, \ldots, X_{r_k}, X_{z_{k+1}}, \ldots, X_{z_n}, X_{\bar{z}_{k+1}}, \ldots, X_{\bar{z}_n}$

span $E^{\mathbb{C}}$ (if not then $\dim D$ would be bigger) which is also spanned by $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{2n-k}}$ (over $\mathbb{C}$) and we now claim that the vectors $Y_i = \frac{\partial}{\partial x^{2n-k+i}}$ $(1 \leq i \leq k)$ satisfy the conditions of the lemma.

Using the coordinates $(x^{k+1}, \ldots, x^{2n})$ on $\pi(U) \subset M/D$ it is obvious that $\pi_* Y_i$ depends only on $\pi(m)$. Now consider on M:

$$dx^j(X_{z_p}) = \omega(X_{z_p}, X_{x^j}) = [z_p, x^j] \qquad k < j \leq 2n\,;$$

x^j constant along $D \Rightarrow X_{x^j} \in E$, $X_{z_p} \in E^{\mathbb{C}} \Rightarrow$ (E is involutive) $X_{[z_p,x^j]} = [X_{z_p}, X_{x^j}] \in E^{\mathbb{C}} \iff [z_p, x^j]$ is constant along D, hence:

$$X_{z_p} = \sum_{j=1}^{2n} (dx^j(X_{z_p})) \frac{\partial}{\partial x^j} \Rightarrow \pi_* X_{z_p} = \sum_{j=k+1}^{2n} dx^j(X_{z_p}) \frac{\partial}{\partial x^j}$$

and the coefficients of $\frac{\partial}{\partial x^j}$ depend only on $\pi(m)$ $(\forall m \in U)$.

Because π is a submersion, because $\pi_*(X_{r_i}) = 0$ and because $(X_{r_i}, X_{z_j}, X_{\bar{z}_j}, Y_i)$ is a basis of $T_m M^{\mathbb{C}}$ it follows that condition (i) is satisfied.

In order to prove condition (ii) we observe that both ε_ω and ω^{n-k} are powers of ω, so it suffices to show that ω applied to any two vectors of the argument is constant on the leaves of D. Therefore consider:

(a) $\omega(X_{r_i}, X_{r_j}) = \omega(X_{r_i}, X_{z_j}) = \omega(X_{r_i}, X_{\bar{z}_j}) = \omega(X_{z_i}, X_{z_k}) = \omega(X_{\bar{z}_i}, X_{\bar{z}_j}) = 0$ because P and $\bar{P}$ are isotropic.

(b) $X_{z_i}, X_{\bar{z}_j} \in E^{\mathbb{C}} \Rightarrow X_{[z_i,\bar{z}_j]} = [X_{z_i}, X_{\bar{z}_j}] \in E^{\mathbb{C}} \iff [z_i, \bar{z}_j] = \omega(X_{z_i}, X_{\bar{z}_j})$ is constant along D.

(c) $\omega(Y_i, X_{z_j}) = \frac{\partial z_j}{x^{2n-k+i}}$ which is constant along D because z_j is $(X_{z_j} \in E^{\mathbb{C}})$.

QED

<u>LEMMA</u> (Newlander-Nirenberg in a special form): if J is a complex structure on a manifold X $(\dim X = 2n)$ such that for each $x_o \in X$ $\exists U$ neighbour-

hood of x_o and $\exists \alpha_1, \ldots, \alpha_n$ complex 1-forms on U which satisfy

(a) $\{\alpha_i\}$ are independent over $\mathbb{C}$ on U

(b) α_i is of type (1,0) on U

(c) α_i is closed.

Then J is an integrable complex structure. (see for instance [Hörmander])

LEMMA: $E_m = \{\mathrm{Re}\, v \mid v \in P_m\} = \{\mathrm{Im}\, v \mid v \in P_m\}$.

PROPOSITION: *suppose* L *is a leaf of* $\tilde{E}$ *in* M/D *then:*

(i) *the map* $J : T_x L \to T_x L$ *defined by*

$$w \in T_x L, \quad \pi(m) = x, \quad v \in P_m : \pi_* \mathrm{Re}\, v = w$$
$$\Rightarrow \quad Jw = J\pi_* \mathrm{Re}\, v = \pi_* \mathrm{Im}\, v$$

is an integrable complex structure on L

(ii) *if* $X_{r_1}, \ldots, X_{r_k}, X_{z_{k+1}}, \ldots, X_{z_n}$ span P *(locally) such that* $X_{r_1}, \ldots, X_{r_k}$ *span* D *then the functions* $z_{k+1}, \ldots, z_n$ *form, when restricted to* L, *a (local) system of complex coordinates for the complex manifold* L.

PROOF: in order to avoid confusion, we have to be very careful, so we denote by $j : L \hookrightarrow M/D$ the canonical injection of the leaf L in M/D. As we have seen earlier, for $m_o \in M$, $x_o = \pi(m_o) \in j(L)$ there exists a neighbourhood U of m_o with coordinates $(x^1, \ldots, x^{2n})$ such that on U : D is spanned by $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^k}\right\}$ and E is spanned by $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{2n-k}}\right\}$. Furthermore, $\pi(U) = V$ is a neighbourhood of x_o in M/D with coordinates $(x^{k+1}, \ldots, x^{2n})$ such that $\left\{\frac{\partial}{\partial x^{k+1}}, \ldots, \frac{\partial}{\partial x^{2n-k}}\right\}$ span $\tilde{E}$ on V, hence there exists a neighbourhood W of x_o in L with coordinates $(x^{k+1}, \ldots, x^{2n-k})$ such that $j(W) = V \cap \{x^i = x^i_o,\ 2n-k < i \leqslant 2n\}$ and moreover: j_* is an isomorphism between TW and $\tilde{E}|_{j(W)}$.

The tactic of the proof is as follows: we first use (ii) to define the complex structure J and we then prove that J is defined in a "coordinate independent" way, and simultaneously that (i) is a correct definition.

Suppose that $X_{r_1},\dots,X_{r_k}, X_{z_{k+1}},\dots,X_{z_n}$ are hamiltonian vector fields on U which span P such that $X_{r_1},\dots,X_{r_k}$ span D, then $X_{z_j} \in P \subset E^{\mathbb{C}}$ hence z_j is constant along D so there exist functions $\tilde{z}_j$ on V: $z_j = \tilde{z}_j \circ \pi$. We denote by a_j and b_j the real and imaginary part of $\tilde{z}_j : \tilde{z}_j = a_j + ib_j$ ($\Rightarrow a_j, b_j : V \to \mathbb{R}$) and we observe that since $X_{r_1},\dots,X_{r_k}$, $X_{z_{k+1}},\dots,X_{z_n}$ span P it follows that the set $\{X_{r_i}, X_{a_j\circ\pi}, X_{b_j\circ\pi}\}$ spans E. Now $X_{r_i} \in D \Rightarrow r_i$ is constant along $E \Rightarrow r_i$ is a function on U independent of the coordinates $x^1,\dots,x^{2n-k}$. It follows that $\{dx^{k+1},\dots,dx^{2n}\}$ span the same space as $\{dr_i, da_j, db_j\}$ and that $\{dx^{2n-k+1},\dots,dx^{2n}\}$ span the same space as $\{dr_1,\dots,dr_k\}$ hence $\{j^*da_j, j^*db_j\}$ is a basis of T^*L on W. Since we know that $\{j^*da_\ell, j^*db_\ell\}$ span T^*L, we can define a map J with respect to this basis: suppose $\xi \in T_xL (x\in W)$ then $J\xi$ is defined by the equations:

$$(j^*da_\ell)(J\xi) = -(j^*db_\ell)(\xi) \qquad k < \ell \leq 2n-k$$
$$(j^*db_\ell)(J\xi) = (j^*da_\ell)(\xi)$$

or equivalently by the real and imaginary parts of the equations

$$(j^*d\tilde{z}_\ell)(J\xi) = i(j^*d\tilde{z}_\ell)(\xi)$$

which shows that $j^*d\tilde{z}_\ell$ is a 1-form on W of type $(1,0)$. Since $\{j^*d\tilde{z}_\ell\}$ is an independent set of 1-forms of type $(1,0)$ which are exact (hence closed) and since their number is half the dimension of L, we can apply the lemma and conclude that J is an integrable complex structure and that $j^*\tilde{z}_{k+1},\dots,j^*\tilde{z}_n$ form a system of complex coordinates on L (N.B. we leave it to the reader to prove that J is a complex structure, i.e. $J^2 = -\mathrm{id}$).

Finally, we have to show that this J coincides with the definition as given in the proposition. Suppose $x \in W \subset L$, $\xi \in T_xL$,

$m \in U : \pi(m) = j(x) \Rightarrow \exists \eta \in \widetilde{E}_{j(x)} : j_*\xi = \eta \Rightarrow \exists w \in E_m : \pi_* w = \eta \Rightarrow$
$\Rightarrow \exists v \in P_m : w = \mathrm{Re}\, v$ hence $j_*\xi = \pi_* \mathrm{Re}\, v$, and we consider $\xi' \in T_x L : \xi' =$
$= j_*^{-1} \pi_* \mathrm{Im}\, v$. Since $X_{z_\ell} \in P_m$ and $v \in P_m$:

$$\omega(v, X_{z_\ell}) = 0 \iff dz_\ell(\mathrm{Im}\, v) = i\, dz_\ell(\mathrm{Re}\, v)$$

and then:

$$\begin{aligned}(j^* d\tilde{z}_\ell)(J\xi) &= i(j^* d\tilde{z}_\ell)(\xi) = i\, d\tilde{z}_\ell(j_*\xi) = i\, d\tilde{z}_\ell(\pi_* \mathrm{Re}\, v) \\ &= i\, dz_\ell(\mathrm{Re}\, v) = dz_\ell(\mathrm{Im}\, v) = d\tilde{z}_\ell(\pi_* \mathrm{Im}\, v) \\ &= (j^* d\tilde{z}_\ell)(\xi')\end{aligned}$$

hence $J\xi = \xi'$. QED

PROPOSITION: *let* $z : U \subset M \to \mathbb{C}$ *then* $X_z \in P$ *if and only if* $\exists \tilde{z} : V \subset M/D \to \mathbb{C} : z = \tilde{z} \circ \pi$ *and* $\tilde{z}$ *is holomorphic when restricted to a leaf of* $\widetilde{E}$.

PROOF: $\Longrightarrow$ $X_z \in P \subset E^{\mathbb{C}} \Rightarrow z$ constant along $D \Rightarrow \exists \tilde{z} : z = \tilde{z} \circ \pi$;
$\forall v \in P : \omega(v, X_z) = 0 \iff \forall v \in P : dz(\mathrm{Im}\, v) = i\, dz(\mathrm{Re}\, v)$ hence $d\tilde{z}(J\pi_* \mathrm{Re}\, v) =$
$= d\tilde{z}(\pi_* \mathrm{Im}\, v) = dz(\mathrm{Im}\, v) = i\, dz(\mathrm{Re}\, v) = i\, d\tilde{z}(\pi_* \mathrm{Re}\, v)$ so $d\tilde{z}$ is of type $(1,0)$ on the leaves of $\widetilde{E}$ hence $\tilde{z}$ is holomorphic when restricted to the leaves of $\widetilde{E}$.

$\Longleftarrow$ let $v \in P$ then, since $d\tilde{z}$ is of type $(1,0)$ on the leaves of $\widetilde{E}$, we have: $\omega(v, X_z) = dz(v) = d\tilde{z}(\pi_* \mathrm{Re}\, v) + i\, d\tilde{z}(\pi_* \mathrm{Im}\, v) = d\tilde{z}(\pi_* \mathrm{Re}\, v) + i\, d\tilde{z}(J\pi_* \mathrm{Re}\, v) =$
$= d\tilde{z}(\pi_* \mathrm{Re}\, v) + i^2\, d\tilde{z}(\pi_* \mathrm{Re}\, v) = 0$ hence, because P is maximal isotropic, $X_z \in P$. QED

9 QUANTIZATION II

In this section we trace the consequences of the use of complex vector fields throughout the quantization-procedure; when necessary we will give adapted definitions and proofs. In the first step, i.e. the construction of the complex line-bundle L with connection ∇ ($\operatorname{curv}\nabla = \omega/\hbar$) and compatible inner product, in this step the use of complex vector fields is evident except (as already noted) that in the compatibility formula there appears a complex conjugation:

$$\zeta(s_1,s_2) = (\nabla_{\bar{\zeta}}\, s_1,s_2) + (s_1,\nabla_{\zeta}\, s_2)$$

which arises from the fact that the inner product is antilinear in the first coordinate.

The next step is the introduction of a complex polarization as described in the previous section, but then we are in "trouble": a real polarization D was needed to define a generalized configuration space M/D of dimension n (if $\dim M = 2n$) and to identify sections which are covariant constant along D with functions on M/D (although only locally). The problem how to integrate over M/D was solved by the introduction of $\frac{1}{2}$-densities; to incorporate these $\frac{1}{2}$-densities in a nice way in the wave functions, we needed the concept of a $-\frac{1}{2}$-D-density on M, but that was a technical question. However, in the case of a complex (admissible) polarization we do have an associated real distribution (also called D to confuse things) but not necessarily of dimension n and moreover, if $\dim D < n$ then M/D exists, but not a nice interpretation of $\frac{1}{2}$-densities on M/D as objects on M, so what do we do? ..... We trust our good luck!! We carry out the the final quantization procedure as described in section 6 and we make the obvious extensions to complex vector fields. In the real case the formula which defines the inner product of two sections of QB which are covariantly constant along

D, this formula is based upon the correspondence between $\frac{1}{2}$-densities on M/D and $-\frac{1}{2}$-D-densities on M. In the complex case, $-\frac{1}{2}$-D-densities change into $-\frac{1}{2}$-P-densities and these cannot be identified with $\frac{1}{2}$-densities on M/D ($D^{\mathbb{C}} = P \cap \bar{P}$); however, by a slight modification of our formula, the inner product turns out as a density on M/D. This suggests that sections of QB which are covariantly constant along P depend upon 2n-k independent parameters (since their inner product resembles a function on M/D and $\dim M/D = 2n-k$), but we will show that such sections can be identified locally with functions on M/D which are holomorphic when restricted to the leaves of $\tilde{E}$, hence they actually depend upon n independent parameters: n-k complex ones (n-k = complex dimension of the leaves of $\tilde{E}$) and k real ones (the rest), which is equivalent to saying that elements of our Hilbert space can be identified (although only locally) with functions of n independent variables. Consequently, a complex polarization also satisfies our goal: to derive a Hilbert space whose elements resemble functions of n variables instead of functions of 2n variables (as is the case in prequantization).

After this lengthy introduction we proceed with the description of the quantization-procedure; the reader is urged strongly to compare this section with section 6 to convince himself that all we do is just a "straightforward" generalization to the complex case.

We start with the construction of the bundle $\hat{R}$ (the analogue of R in the case of real polarizations): $\hat{R}$ is the bundle of all complex frames of P, i.e. the fibre $\hat{R}_m$ consists of all complex frames of P_m. Since all frames of P_m can be indexed by $GL(n,\mathbb{C})$ (because $\dim_{\mathbb{C}} P_m = n$) the bundle $\hat{R}$ is a principal $GL(n,\mathbb{C})$ bundle over M; on the other hand, $\hat{R}$ is a subbundle of $F^nM^{\mathbb{C}}$, the bundle of *all* complex n-frames on M.
A $-\frac{1}{2}$-P-density ν on M is a complex valued function on $\hat{R}$ such that:

$$(\eta) \in R_m,\ g \in GL(n,\mathbb{C}) \Rightarrow \nu((\eta)g) = \nu((\eta)) \cdot |\det g|^{-\frac{1}{2}}$$

and special attention is called for the fact that the absolute value of $\det g$ is used which enables us to define the (positive) square root of it.

The bundle B^P is the bundle over M whose fibre B^P_m consists of all functions $\nu_m : \hat{R}_m \to \mathbb{C}$ satisfying:

$$\nu_m((\eta)g) = \nu_m((\eta))\, |\det g|^{-\frac{1}{2}}, \qquad (\eta) \in \hat{R}_m,\ \ g \in GL(n,\mathbb{C}).$$

Since $\hat{R}$ is a principal $GL(n,\mathbb{C})$ bundle, B^P is a complex line-bundle (a function ν_m is determined completely by its value on an arbitrary but fixed frame), sections of B^P coincide with $-\frac{1}{2}$-P-densities on M and, using partitions of unity, one can prove that B^P is a trivial bundle.

We now have to define the (partial) connection ∇ on the bundle B^P; at a first glance this seems an obvious definition, knowing the construction in the real case, but there are complications. In the real case the connection is defined for vectors in $D = E$; in the complex case we will define the connection for vectors in $E^{\mathbb{C}} = P + \bar{P}$. Suppose $\zeta \in E^{\mathbb{C}}$ is any (complex) vector field in $P + \bar{P}$ and ν any section of B^P, then we have to define a new section $\nabla_\zeta \nu$ of B^P. This $-\frac{1}{2}$-P-density is defined completely by its value on one P-frame, so suppose $m_o \in M$, U a neighbourhood of m_o and $(\eta_1,\ldots,\eta_n)$ vector fields on U which span P such that $(\eta_1,\ldots,\eta_k)$ are (real) hamiltonian vector fields spanning D, then we define $(\nabla_\zeta \nu)(m_o, (\eta_i|_{m_o}))$ by:

$$(\nabla_\zeta \nu)(m_o, (\eta_i|_{m_o})) \cdot |\varepsilon_{\omega,k}(\eta_{k+1},\ldots,\eta_n, \bar{\eta}_{k+1},\ldots,\bar{\eta}_n)|^{\frac{1}{4}} =$$
$$= \zeta_{m_o}[\nu(m, (\eta_i|_m)) \cdot |\varepsilon_{\omega,k}(\eta_{k+1},\ldots,\eta_n, \bar{\eta}_{k+1},\ldots,\bar{\eta}_n)|^{\frac{1}{4}}]$$

where the $2(n-k)$-form $\varepsilon_{\omega,k}$ is defined by

$$\varepsilon_{\omega,k} = \frac{(-1)^{\frac{1}{2}(n-k)(n-k+1)}}{(n-k)!} \cdot \omega^{n-k} \qquad\qquad (\Rightarrow \varepsilon_{\omega,o} = \varepsilon_\omega).$$

If P is real (i.e. $P = D^{\mathbb{C}}$) then $k = n$, $\varepsilon_{\omega,n} = 1$, $E = D$ and we recover the previous definition. However, if $k < n$ then the definition of $\nabla_\zeta \nu$ is quite complicated and one can ask: what happens if we omit the facto $|\varepsilon_{\omega,k}|^{\frac{1}{4}}$? The answer is that this factor is needed to insure that $\nabla_\zeta \nu$ is a well-defined $-\frac{1}{2}$-P-density:

PROPOSITION: *suppose* $(\hat{\eta}_i)$ *is another* P-*frame on* U *satisfying the same conditions such that* $\hat{\eta}_i = \eta_j g_{ji}$, $g(m) \in GL(n,\mathbb{C})$, *then*

$$(\nabla_\zeta \nu)(m_o, (\hat{\eta}_i|_{m_o})) = (\nabla_\zeta \nu)(m_o, (\eta_j|_{m_o})) \cdot |\det g(m_o)|^{-\frac{1}{2}} .$$

PROOF: since both (η) and $(\hat{\eta})$ satisfy the condition that their first k vectors span D, it follows that g can be written as:

$$g = \begin{pmatrix} a & ? \\ \emptyset & b \end{pmatrix}, \quad a \in GL(k,\mathbb{R}) , \quad b \in GL(n-k,\mathbb{C})$$

whence $\nu(m, (\hat{\eta}|_m)) \cdot |\varepsilon_{\omega,k}(\hat{\eta}_{k+1},\ldots,\bar{\hat{\eta}}_n)|^{\frac{1}{4}}$

$$= \nu(m, (\eta|_m)) \cdot |\varepsilon_{\omega,k}(\eta_{k+1},\ldots,\bar{\eta}_n)|^{\frac{1}{4}} \cdot |\det a|^{-\frac{1}{2}}$$

(use that P is isotropic or see the proof of the second next proposition). Since η_i and $\hat{\eta}_i$ $(i = 1,\ldots,k)$ are (locally) hamiltonian vector fields there exist functions $f_i, \hat{f}_i$ such that $\eta_i = X_{f_i}$, $\hat{\eta}_i = X_{\hat{f}_i}$ $(i = 1,\ldots,k)$ and we have $X_{\hat{f}_i} = X_{f_j} a_{ji}$ hence for each $X_s \in E$ (s a real function): $[X_s, X_{f_j}] = X_{\omega(X_s, X_{f_j})} = 0$ (because $X_{f_j} \in D$); in the same way $[X_s, X_{\hat{f}_i}] = 0$, so it follows that $(X_s a_{ji}) X_{f_j} = 0$, whence $X_s a_{ji} = 0$ (because the X_{f_j} are independent). Since the X_s span E, it follows that $\forall \zeta \in E : \zeta_{m_o} a_{ji} = 0$. Because $\det a$ is a polynomial in the a_{ji} and because the sign of $\det a$ is locally constant (a is a *real* matrix with non-zero determinant) it follows that $\zeta_{m_o} |\det a| = 0$ and from this one deduces the proposition.

QED

REMARK 1: if we had omitted the correcting factor $|\varepsilon_{\omega,k}|^{\frac{1}{4}}$, we would have had troubles when we wish to prove that the *absolute value* of $\det g$ is constant along $E^{\mathbb{C}}$ or P, so we see that the use of the absolute value in the definition of $-\frac{1}{2}$-P-densities prohibits a nice definition of the connection in B^P!

REMARK 2: the correction factor $|\varepsilon_{\omega,k}|^{\frac{1}{4}}$ is such that the combination

$$\nu(\eta_i) \cdot |\varepsilon_{\omega,k}(\eta_{k+j},\bar{\eta}_{k+j})|^{\frac{1}{4}}$$

depends only on the vectors $\eta_1, \dots, \eta_k$ spanning D, so one might say that $\nu|\varepsilon_{\omega,k}|^{\frac{1}{4}}$ is a function on the D-frames satisfying the $-\frac{1}{2}$-density relation, i.e. if $(\eta_1,\dots,\eta_k)$ span D and if $g \in GL(k,\mathbb{R})$ then

$$(\nu \cdot |\varepsilon_{\omega,k}|^{\frac{1}{4}})(\eta_i g) = (\nu \cdot |\varepsilon_{\omega,k}|^{\frac{1}{4}})(\eta_i) \cdot |\det g|^{-\frac{1}{2}}$$

The interested reader is referred to [Woodhouse §5.10] for a more detailed description of this relationship.

COROLLARY: $\nabla_\zeta \nu$ *is a well-defined* $-\frac{1}{2}$-P-*density.*

We now state with a reference to section 5 and without proof:

PROPOSITION: ∇ *possesses all properties of a flat (partial, i.e. along* $E^{\mathbb{C}}$*) connection on the bundle* B^P.

The next step is to define the line-bundle QB over M as $QB = L \otimes B^P$ with its partial connection ∇ defined by

$$\nabla_\zeta(s \otimes \nu) = (\nabla_\zeta s) \otimes \nu + s \otimes (\nabla_\zeta \nu).$$

We then have to define an inner product on sections of QB which are covariant constant along P; this we will do in such a way that (ψ_1,ψ_2) turns out as a density on M/D ($D^{\mathbb{C}} = P \cap \bar{P}$). As in the case of a real polarization, we do it in two steps: suppose that $\psi_i = s_i \otimes \nu_i$ (i = 1,2) are two arbitrary sections of QB (it does not matter if the representation $\psi_i = s_i \otimes \nu_i$ holds only locally), suppose $m \in M$, $x = \pi(m) \in M/D$ and suppose $(\zeta_1,\ldots,\zeta_n, \xi_1,\ldots,\xi_n)$ is a basis of $T_m M^{\mathbb{C}}$ satisfying the conditions:

$$(\zeta_1,\ldots,\zeta_n) \text{ is a basis of } P_m \quad (\Longleftrightarrow (\zeta_i) \in \hat{R}_m)$$

$$(\zeta_1,\ldots,\zeta_k) \text{ is a basis of } P_m \cap \bar{P}_m = D_m^{\mathbb{C}}$$

then $\pi_*(\zeta_{k+1},\ldots,\zeta_n, \xi_1,\ldots,\xi_n)$ is a basis of $T_x(M/D)^{\mathbb{C}}$ (π is a submersion!) and we now define a function $(\psi_1,\psi_2)_m$ on $F_x^{2n-k}(M/D)^{\mathbb{C}}$ (i.e. on the space of all basis of $T_x(M/D)^{\mathbb{C}}$) by:

$$(\psi_1,\psi_2)_m(\pi_*(\zeta_{k+1},\ldots,\zeta_n, \xi_1,\ldots,\xi_n) = (s_1,s_2)(m) \cdot \overline{\nu_1(m, (\zeta))} \cdot \nu_2(m, (\zeta)) \cdot$$
$$\cdot \, |\varepsilon_{\omega,k}(\zeta_{k+1},\ldots,\zeta_n, \bar{\zeta}_{k+1},\ldots,\bar{\zeta}_n)|^{\frac{1}{2}} \cdot |\varepsilon_\omega(\zeta_1,\ldots,\zeta_n, \xi_1,\ldots,\xi_n)|$$

Since the basis $\pi_*(\zeta_{k+1},\ldots,\xi_n)$ is not an arbitrary basis of $T_x(M/D^{\mathbb{C}}$, $(\psi_1,\psi_2)_m$ is not defined on the whole of $F_x^{2n-k}(M/D)^{\mathbb{C}}$, but it satisfies the relations of a 1-density:

PROPOSITION: $(\psi_1,\psi_2)_m$ *defines a unique* 1-*density on* $F_x^{2n-k}(M/D)^{\mathbb{C}}$.

PROOF: since a 1-density is determined by its value on one frame, it is sufficient to show that the values of $(\psi_1,\psi_2)_m$ at different frames are related in the correct way. So suppose $(\hat{\zeta}_1,\ldots,\hat{\zeta}_n, \hat{\xi}_1,\ldots,\hat{\xi}_n)$ is another basis of $T_m M^{\mathbb{C}}$ satisfying the same conditions, then they are related by a matrix $h \in GL(2n,\mathbb{C})$ of the form:

$$h = \begin{pmatrix} a & ? & ? \\ \emptyset & b & \\ & \emptyset & c \end{pmatrix} \quad , \quad \begin{array}{l} a \in GL(k,\mathbb{C}) \\ b \in GL(n-k,\mathbb{C}) \\ c \in GL(n,\mathbb{C}) \end{array}$$

$$(\hat{\zeta}_i,\hat{\xi}_i) = (\zeta_j,\xi_j) \cdot h$$

it follows that the bases $\pi_*(\hat{\zeta},\hat{\xi})$ and $\pi_*(\zeta,\xi)$ are related by a matrix $g \in GL(2n-k,\mathbb{C})$ of the form:

$$g = \begin{pmatrix} b & ? \\ \emptyset & c \end{pmatrix} , \qquad \det g = \det b \cdot \det c$$

$$\pi_*(\hat{\zeta},\hat{\xi}) = \pi_*(\zeta,\xi) \cdot g .$$

Since ε_ω is a volume form on M we have:

$$|\varepsilon_\omega(\hat{\zeta}_i,\hat{\xi}_i)| = |\varepsilon_\omega(\zeta_j,\xi_j)| \cdot |\det h| ,$$

since ν_i is a $-\frac{1}{2}$-P-density we have

$$\nu_i(\hat{\zeta}_j) = |\det a \cdot \det b|^{-\frac{1}{2}} \nu_i(\zeta_j)$$

and finally since $\hat{\zeta}_j = b_{ji}\zeta_i + v_j$ where $v_j \in P \cap \bar{P}$ and since P is isotropic, it follows that

$$|\varepsilon_{\omega,k}(\hat{\zeta},\bar{\hat{\zeta}})|^{\frac{1}{2}} = |\det b \cdot \det \bar{b}|^{\frac{1}{2}} \cdot |\varepsilon_{\omega,k}(\zeta,\bar{\zeta})|^{\frac{1}{2}}$$

so in total we get:

$$(\psi_1,\psi_2)_m(\pi_*(\hat{\zeta},\hat{\xi})) = (\psi_1,\psi_2)_m(\pi_*(\zeta,\xi)) \cdot |\det h| \cdot |\det b| \cdot \overline{|\det a \cdot \det b|^{-\frac{1}{2}}} \cdot |\det a \cdot \det b|^{-\frac{1}{2}}$$

$$\Longleftrightarrow \quad (\psi_1,\psi_2)_m(\pi_*(\hat{\zeta},\hat{\xi})) = (\psi_1,\psi_2)_m(\pi_*(\zeta,\xi)) \cdot |\det g|$$

which is the correct transformation of a 1-density at $x \in M/D$. QED

If we compare this formula of $(\psi_1,\psi_2)_m$ with the formula in the case of a real polarization(see section 6) then two differences can be detected: we have been more careful in the choice of the basis of T_mM, due to the fact that P_m does not coincide with the kernel of π_* ; this implied that

$(\psi_1,\psi_2)_m$ was a priori not defined on the whole of $F_x^{2n-k}(M/D)^{\mathbb{C}}$. Secondly, there appears an extra factor $|\varepsilon_{\omega,k}|^{\frac{1}{2}}$ in the definition. "This" factor already appeared in the definition of the connection on B^P (at least the square root of it, but here we have two $-\frac{1}{2}$-P-densities ν_1 and ν_2) and is needed to insure that $(\psi_1,\psi_2)_m$ is indeed a density at $x \in M/D$. It should be noted that our definition of $(\psi_1,\psi_2)_m$ is in agreement with [Sniatycki] and differ from [Simms] by a factor $(n-k)!$, a factor which appears in the definition of $\varepsilon_{\omega,k}$.

Having proved that $(\psi_1,\psi_2)_m$ defines a density at $x = \pi(m) \in M/D$ we now want to prove that, if ψ_1 and ψ_2 are covariant constant along P, then the density $(\psi_1,\psi_2)_m$ is independent of the choice of $m \in \pi^{-1}(x)$, which should imply that ψ_1 and ψ_2 define a unique density on M/D denoted by (ψ_1,ψ_2)

PROPOSITION: *if* ψ_1 *and* ψ_2 *are covariant constant along* P *then the density* $(\psi_1,\psi_2)_m$ *is constant on the leaves of* D, *i.e.* ψ_1 *and* ψ_2 *define a unique density on* M/D.

PROOF: by definition of a complex polarization there exist in a neighbourhood U of $m_o \in M$ hamiltonian vector fields $X_{z_1}, \ldots, X_{z_n}$ which span P (on U) and such that $X_{z_1}, \ldots, X_{z_k}$ span D, i.e. $z_1, \ldots, z_k$ are real functions. In a lemma of the previous section we proved that there exist vector fields $Y_1, \ldots, Y_k$ on U (or possibly a smaller neighbourhood of m_o) such that:

(i) $\pi_*(X_{z_{k+1}},\ldots,X_{z_n}, X_{\bar{z}_{k+1}},\ldots,X_{\bar{z}_n}, Y_1,\ldots,Y_k)$ is a basis of $T_{\pi(m)}(M/D)^{\mathbb{C}}$ which depends only on $\pi(m)$

(ii) $\varepsilon_\omega(X_{z_1},\ldots,X_{z_n}, X_{\bar{z}_{k+1}},\ldots,X_{\bar{z}_n}, Y_1,\ldots,Y_k)$ is a function on U which is constant on the leaves of D.

Now $\phi(m) := (\psi_1,\psi_2)_m(\pi_*(X_{z_{k+1}},\ldots,X_{z_n}, X_{\bar{z}_{k+1}},\ldots,X_{\bar{z}_n}, Y_1,\ldots,Y_n)$ is a function on U whose value $\phi(m)$ is the value of the density $(\psi_1,\psi_2)_m$ at the given frame of $\pi(m)$. Since this frame does depend on $\pi(m)$ only, it follows that if ϕ is constant on the leaves of D, then the density $(\psi_1,\psi_2)_m$ depends only on $\pi(m)$, which is just the claim of our proposition. So suppose $\xi \in D_m$ then:

$$(\xi\phi)(m) = |\varepsilon_\omega| \cdot \xi[(s_1,s_2)(m)\, \overline{\nu_1(m, (X_{z_i}|_m))} \cdot \nu_2(m, (X_{z_i}|_m)) \cdot |\varepsilon_{\omega,k}|^{\frac{1}{2}}]$$

because of property (ii) above!

$$= (\nabla_\zeta\psi_1,\psi_2)_m(\pi_*(X_z, X_{\bar{z}}, Y)) + (\psi_1,\nabla_\xi\psi_2)_m(\pi_*(X_z, X_{\bar{z}}, Y)) =$$

$$= 0$$

because $\xi \in D \subset P$ and because ψ_1 and ψ_2 are covariant constant along P. This proves that $(\psi_1,\psi_2)_m$ is locally constant on the leaves of D; by connectedness of the leaves of D the proposition follows. QED

REMARK: it is rather hard to prove that given a section $\psi = s \otimes \nu$ which is covariant constant along P, then there exists a function f on M (locally) such that $f \cdot s$ and $\frac{1}{f} \cdot \nu$ are both covariant constant along P. We therefore never used the fact that sections ψ of QB which are covariant constant along P admit representations $\psi = s \otimes \nu$ in which both s and ν are covariant constant along P. However, such representations do exist (at least locally) as can be seen in the following explicit construction: suppose $m_o \in M$ and $(\eta_1,\ldots,\eta_n)$ vector fields on a neighbourhood U of m_o which span P such that $(\eta_1,\ldots,\eta_k)$ are (real) hamiltonian vector fields spanning D. Then the $-\frac{1}{2}$-P-density ν_o defined by

$$\nu_o(m, (\eta_i|_m)) = |\varepsilon_{\omega,k}(\eta_{k+1},\ldots,\eta_n, \bar{\eta}_{k+1},\ldots,\bar{\eta}_n)|^{-\frac{1}{4}}$$

is a nowhere vanishing $-\frac{1}{2}$-P-density on U which is covariant constant along P. To prove that ν_o is non-vanishing, we observe that there exist vector fields $Y_1, \ldots, Y_k$ such that $(\xi_1,\ldots,\xi_{2n}) :=$ $:= (\eta_1,\ldots,\eta_n, \bar{\eta}_{k+1},\ldots,\bar{\eta}_n, Y_1,\ldots,Y_k)$ constitutes a basis of $T_m M^{\mathbb{C}}$ in a neighbourhood of m_o. By isotropy of P it follows that:

$$\omega_{ij} = \omega(\xi_i,\xi_j) = \begin{array}{c} 1\to k \quad \to 2n-k \quad \to 2n \\ \left(\begin{array}{c:c:c} \emptyset & \emptyset & g \\ \hdashline \emptyset & \tilde{\omega} & ? \\ \hdashline -g & -? & ?? \end{array}\right) \end{array} .$$

Since ω is nondegenerate it follows that $\det\tilde{\omega} \neq 0$ which implies that ν_o (which is related to $\tilde{\omega}$) is non-vanishing. Since ν_o is non-vanishing on U, each section ψ of QB admits on U a representation $\psi = s \otimes \nu_o$ and hence if ψ is covariant constant along P, so is s.

From the construction above one can deduce that if D is globally spanned by (locally) hamiltonian vector fields, then ν_o is a global non-vanishing $-\frac{1}{2}$-P-density which is covariant constant along P.

With the construction of a (correctly defined) density (ψ_1,ψ_2) on M/D associated to two sections ψ_1, ψ_2 of QB which are covariant constant along P, we now can proceed with the construction of the Hilbert space $\mathcal{H}$. First we define a prehilbert space PH by

$$PH = \{\psi: M \to QB \mid (\forall\zeta \in P : \nabla_\zeta \psi = 0) \wedge \int_{M/D} (\psi,\psi) < \infty\} ,$$

a prehilbert space consisting of (C^∞) sections of QB, which are covariant constant along P with finite integral over M/D, on which the inner product is defined by

$$<\psi_1,\psi_2> = \int_{M/D} (\psi_1,\psi_2) , \qquad \psi_1, \psi_2 \in PH .$$

The Hilbert space $\mathcal{H}$ of the quantum mechanical description just is the Hilbert space associated to the prehilbert space PH. Concerning the elements of $\mathcal{H}$ (more precisely: of PH) we can state the following proposition:

PROPOSITION: *suppose* ψ_o *is a section of* QB, *covariant constant along* P, *suppose* ψ_o *is non-vanishing on the neighbourhood* $U \subset M$ *and let* ψ *be any section of* QB *then:* ψ *is covariant constant along* P *(on* U*)* *if and only if* $\exists \tilde{f} : M/D \to \mathbb{C}, \tilde{f}$ *holomorphic on the leaves of* $\tilde{E}$ *such that* $\psi = (\tilde{f} \circ \pi) \cdot \psi_o$ *(on* U*).*

PROOF: because QB is a line-bundle and because ψ_o is non-vanishing on U, there exists a function $f: U \to \mathbb{C}$ such that $\psi = f \cdot \psi_o$ on U. Hence on U we have: $\forall \zeta \in P : \nabla_\zeta \psi = 0 \iff \forall \zeta \in P : \zeta f = 0 \iff \forall \zeta \in P : \omega(\zeta, X_f) = 0 \iff$ $\iff X_f \in P \iff \exists \tilde{f},\ f = \tilde{f} \circ \pi$ and f holomorphic on the leaves of $\tilde{E}$ (the last equivalence was proved in the previous section). QED

The last step in the quantization procedure is the definition of the quantizable observables; just as in the case of a real polarization we can define the operator L_ζ on $-\frac{1}{2}$-P-densities on M: for real vector fields ζ satisfying $[\zeta, P] \subset P$ with associated flow ρ_t on M and for ν a $-\frac{1}{2}$-P-density on M we define:

$$(L_\zeta \nu)(m, (\eta)) = \frac{d}{dt}\Big|_{t=0} \nu(\rho_t m, \rho_{t*}(\eta))$$

and, as before, one can prove that $L_\zeta \nu$ is a $-\frac{1}{2}$-P-density and that L_ζ has the properties of a Lie derivative (except that it is defined for a special class of vector fields), more specifically:

PROPOSITION: *suppose* ζ *and* ζ' *are real vector fields on* M *such that* $[\zeta, P] \subset P$ *and* $[\zeta', P] \subset P$ $(\Rightarrow [[\zeta, \zeta'], P] \subset P)$, *suppose* g *is a complex* *function on* M *and* ν, ν' *two* $-\frac{1}{2}$-P-*densities on* M *then:*

(i) $L_\zeta(g\nu) = g\, L_\zeta \nu + (\zeta g)\nu\,, \quad L_\zeta(\nu+\nu') = L_\zeta \nu + L_\zeta \nu'$

(ii) $L_\zeta L_{\zeta'}\nu - L_{\zeta'} L_\zeta \nu = L_{[\zeta,\zeta']}\nu$

(iii) *if* f *is a real function on* M *such that* $X_f \in P$ *(hence* $X_f \in D$) *then:*

$$L_{X_f} \nu = \nabla_{X_f} \nu\,.$$

With the definition of L_ζ on the set of $-\frac{1}{2}$-P-densities we can define the set of quantizable observables and the associated operators on H : an observable f (i.e. $f\colon M \to \mathbb{R}$) is quantizable iff $[X_f,P] \subset P$; if f is quantizable and if $\psi \in PH \subset H$ then

$$\delta(f)\psi = \delta(f)s \otimes \nu = (-i\hbar \nabla_{X_f} s + fs) \otimes \nu - i\hbar s \otimes (L_{X_f} \nu)$$

which becomes $\delta(f)\psi = -i\hbar \nabla_{X_f} \psi + f\psi$ if $X_f \in P$. This definition is valid on QB and it remains to show that $\delta(f)$ maps PH into PH and moreover, we claim (as before) that $\delta(f)$ is essentially self-adjoint if X_f is complete. Of these statements, we will only prove that if ψ is covariant constant along P, then $\delta(f)\psi$ too; the rest of our claims can be proved completely analogous to the case of a real polarization, so we leave that part to the reader. In order to prove our part, we need two lemma's and a proposition; the first lemma also is useful to prove part (iii) of the previous proposition.

<u>LEMMA</u>: if f is a quantizable observable, ρ_t the flow associated to X_f and if $(\eta_1,\ldots,\eta_n)$ are (locally) hamiltonian vector fields which span P such that $(\eta_1,\ldots,\eta_k)$ span D ($\Rightarrow (\eta_1,\ldots,\eta_k)$ are real) then:

(i) $\rho_{t*}(\eta_1|_m,\ldots,\eta_n|_m) = (\eta_1|_{\rho_t m},\ldots,\eta_n|_{\rho_t m}) \cdot g_t(m)\,, \quad g_t(m) \in GL(n,\mathbb{C})$

with $g_t(m) = \begin{pmatrix} a_t(m) & ? \\ \emptyset & b_t(m) \end{pmatrix}$, $a_t(m) \in GL(k,\mathbb{R})$, $b_t(m) \in GL(n-k,\mathbb{C})$

(ii) the coefficients of $a_t(m)$ are constant along E,

(iii) the coefficients of $b_t(m)$ are constant along D.

PROOF: $\eta_i = X_{e_i} \Rightarrow \rho_{t*} X_{e_i}\big|_m = X_{e_i \circ \rho_{-t}}\big|_{\rho_t m} \Rightarrow \rho_{t*}\eta_i$ is (locally) hamiltonian.

Because $X_{e_i \circ \rho_{-t}}$, $X_{e_j} \in P$ which is isotropic it follows that:

$$[\rho_{t*} X_{e_i}, X_{e_j}] = [X_{e_i \circ \rho_{-t}}, X_{e_j}] = X_{\omega(X_{e_i \circ \rho_{-t}}, X_{e_j})} = 0$$

$$\Rightarrow \ 0 = [X_{e_k} g_{ki\,t}(m), X_{e_j}] = -(X_{e_j} g_{ki\,t}(m))\, X_{e_i}$$

$$\Rightarrow \ X_{e_j} g_{ki\,t} = 0 .$$

Since $(X_{e_i})_{i=1}^k$ and $(\rho_{t*} X_{e_i})_{i=1}^k$ span D ($\rho_{t*}\eta_i$ is a real vector field in P for $1 \leqslant i \leqslant k$) condition (i) follows; since a_{ki} is real the condition $X_{e_j} a_{ki} = 0$ implies that the real and imaginary part yields zero separately, hence a_{ki} is constant along E. Finally, since X_{e_i}, $1 \leqslant i \leqslant k$ are real it follows that b_{ki} is constant along D. QED

LEMMA: let f be a quantizable observable, η a locally hamiltonian vector field $\eta \in E$ and let ν be any section of B^P, then

$$\nabla_\eta L_{X_f} \nu - L_{X_f} \nabla_\eta \nu = \nabla_{[\eta, X_f]} \nu .$$

PROOF: let $m_o \in M$, U a neighbourhood of m_o such that there exist: (i) a function e on U: $\eta = X_e$ (e is real) and (ii) hamiltonian vector fields $\eta_1, \ldots, \eta_n$ on U which span P where $\eta_1, \ldots, \eta_k$ span D. If we denote by $\hat{\nu}$ the function on U defined by

$$\dot{\nu}(m) = \nu(m, (\eta_i|_m)),$$

if we denote by $\dot{\varepsilon}$ the function on U defined by

$$\dot{\varepsilon}(m) = |\varepsilon_{\omega,k}(\eta_{k+1}|_m,\dots,\eta_n|_m, \bar{\eta}_{k+1}|_m,\dots,\bar{\eta}_n|_m)|^{\frac{1}{4}}$$

and if $g_t(m)$ is the matrix defined in the previous lemma, then:

$$(\nabla_{X_e}\nu)(m_o, (\eta_i|_{m_o})) = X_e|_{m_o}\dot{\nu} + \dot{\nu}(m_o)X_e|_{m_o} \log \dot{\varepsilon}$$

$$(L_{X_f}\nu)(m_o, (\eta_i|_{m_o})) = X_f|_{m_o}\dot{\nu} + \dot{\nu}(m_o)\frac{d}{dt}\Big|_{t=0}(|\det g_t(m_o)|^{-\frac{1}{2}}$$

From these observations it follows that

$$([\nabla_{X_e},L_{X_f}]\nu)(m_o, (\eta_i|_{m_o})) = (\nabla_{[X_e,X_f]}\nu)(m_o, (\eta_i|_{m_o})) +$$

$$+ \dot{\nu}(m_o)X_e|_{m_o}\left\{\frac{d}{dt}\Big|_{t=0}\left(|\det g_t(m))|^{-\frac{1}{2}} - \frac{\dot{\varepsilon}(\rho_t m)}{\dot{\varepsilon}(m)}\right)\right\}.$$

Now we observe that, because $\rho_t^*\omega = \omega$ (X_f is locally hamiltonian)

$$|\varepsilon_{\omega,k}|_m(\eta_j|_m, \bar{\eta}_j|_m)|^{\frac{1}{4}} = |\varepsilon_{\omega,k}|_{\rho_t m}(\rho_{t*}\eta_j|_m, \rho_{t*}\bar{\eta}_j|_m)|^{\frac{1}{4}}$$

hence $\dot{\varepsilon}(\rho_t m) = \dot{\varepsilon}(m)\cdot|\det b_t(m)|^{-\frac{1}{2}}$

so

$$X_e|_{m_o}\frac{d}{dt}\Big|_{t=0}\left[|\det g_t(m)|^{-\frac{1}{2}} - \frac{\dot{\varepsilon}(\rho_t(m)}{\dot{\varepsilon}(m)}\right] =$$

$$= X_e|_{m_o}\frac{d}{dt}\Big|_{t=0}\left[|\det b_t(m)|^{-\frac{1}{2}}\cdot(|\det a_t(m)|^{-\frac{1}{2}} - 1)\right]$$

$$= 0$$

because $a_t(m)$ is constant along $E \ni X_e$ and $b_o(m) = id$, $a_o(m) = id$.

QED

<u>PROPOSITION</u>: *if* f *is a quantizable observable,* ζ *a locally hamiltonian vector field on* M, $\zeta \in E$ *and* ψ *any section of* QB *then:*

$$\nabla_\zeta \delta(f)\psi = \delta(f)\nabla_\zeta\psi - i\hbar\nabla_{[\zeta,X_f]}\psi .$$

PROOF: this can be verified directly using the curvature of the connection ∇ on L and the previous lemma. QED

COROLLARY: *if* f *is a quantizable observable and* ψ *covariant constant along* P, *then* $\delta(f)\psi$ *is covariant constant along* P.

This finishes the quantization procedure in case of a complex admissible polarization; we conclude this section with a summary of this quantization procedure.

In this and previous section we have generalized the notion of a real polarization to a complex polarization and we have studied the consequences of this generalization. A complex polarization P is a complex distribution P of (complex) dimension n on the symplectic manifold (M,ω) with the following local properties:

(i) $\exists z_1,\dots,z_n : X_{z_1},\dots,X_{z_n}$ span P

(ii) $[z_i,z_j] = 0$

(iii) $\dim_{\mathbb{C}}(P\cap\bar{P}) = k$ constant on M

(iv) $\exists w_1,\dots,w_k : X_{w_1},\dots,X_{w_k}$ span $P\cap\bar{P}$

Associated to P we defined two real distributions D and E on M by

$$D^{\mathbb{C}} = P\cap\bar{P}, \quad \dim_{\mathbb{R}} D = k$$

$$E^{\mathbb{C}} = P+\bar{P}, \quad \dim_{\mathbb{R}} E = 2n-k$$

and we showed that both are integrable distributions on M. We called a complex polarization P admissible if the set M/D admits a manifold structure for which $\pi: M \to M/D$ is a submersion, and we henceforth studied admissible complex polarizations only.

At the end of the previous section we showed that the distribution $\tilde{E} = \pi_* E$ is a well-defined foliation of M/D and that the leaves of $\tilde{E}$ admit the structure of a complex manifold such that

$$X_z \in P \iff z \text{ is a function on } M/D\text{, holomorphic on the leaves of } \tilde{E}.$$

The next step in the quantization procedure was the construction of the bundle B^P analogous to the case of a real polarization, while we omitted (necessarily!) the step in which we should identify sections of B^P (called $-\frac{1}{2}$-P-densities) with $\frac{1}{2}$-densities on some n-dimensional manifold. On B^P we defined (for vectors in E) a connection ∇ by the formula

$$(\nabla_\zeta \nu)(m_o, (\eta_o)) = |\varepsilon_{\omega,k}(\eta_j, \bar{\eta}_j)|^{-\frac{1}{4}} \cdot \zeta|_{m_o} \left[\nu(m, (\eta_i)_m) \cdot |\varepsilon_{\omega,k}(\eta_j, \bar{\eta}_j)|^{\frac{1}{4}}\right]$$

where η_i $(i = 1,..,k)$ are (locally) hamiltonian vector fields spanning D, where $\eta_1, \dots, \eta_n$ span P and where $(\eta_i|_{m_o}) = \eta_o$ (note that it is not necessary that the $\eta_{k+1}, \dots, \eta_n$ are locally hamiltonian vector fields).

The subsequent steps in the quantization procedure were the "straightforward" generalizations of the real case: the construction of the bundle $QB = L \otimes B^P$ with its connection ∇, the definition of the inner product on sections of QB which are covariant constant along P:

$$(\psi_1,\psi_2)(\pi_*(X_z, X_{\bar{z}}, Y)) = (s_1,s_2)\overline{\nu_1(X_z)}\nu_2(X_z)|\varepsilon_\omega(X_z, X_{\bar{z}}, Y)| \cdot |\varepsilon_{\omega,k}(X_z, X_{\bar{z}})|^{\frac{1}{2}}$$

This was followed by the definition of the Hilbert space H:

$$H \text{ "=" } \{\psi: M \to QB \mid (\forall \zeta \in P : \nabla_\zeta \psi = 0) \wedge \int_{M/D} (\psi,\psi) < \infty\}$$

$$<\psi_1,\psi_2> = \int_{M/D} (\psi_1,\psi_2),$$

the definition of the "Lie derivative" of a $-\frac{1}{2}$-P-density (for special real vector fields only!) and finally the definition of quantizable observables:

$$f \text{ quantizable} \iff [X_f, P] \subset P$$

$$\delta(f)(s \otimes \nu) = (-i\hbar\nabla_{X_f} + fs) \otimes \nu - i\hbar\, s \otimes (L_{X_f} \nu).$$

10 SOME EXAMPLES II

In this section we will give two examples of (non real) complex polarizations. The first example is the 1-dimensional harmonic oscillator, but with the complex polarization we do not have to omit the origin of $M = \mathbb{R}^2$; the Hilbert space now comes out correctly, however, the energy levels remain the wrong ones: $E_n = n\hbar$ instead of $(n+\frac{1}{2})\hbar$.
The second example is the cotangent bundle of $S^2 : M = T^*S^2$ but (as in the case of the harmonic oscillator with the real polarization) we have to omit the zero section in order to quantize the energy of a free particle on S^2.

EXAMPLE 1: *the* 1-*dimensional harmonic oscillator: holomorphic representation.*
In this example we quantize the symplectic manifold $(\mathbb{R}^2, dp \wedge dq)$ using the polarization $P_{hol} = \mathbb{C}\left(\frac{\partial}{\partial p} + i\frac{\partial}{\partial q}\right) = \mathbb{C} \cdot X_{p+iq}$. This polarization will allow us to quantize the observables $H = \frac{1}{2}(p^2+q^2)$, p and q ; moreover, the Hilbert space does not vanish: in this case we do not need distribution valued sections.

1A. Prequantization: as usual we take $L = M \times \mathbb{C}$ and we use the symplectic potential $\theta = \frac{1}{2}(p\,dq - q\,dp)$ (the reader is urged to calculate the effect if one uses $\theta = p\,dq$ instead of our choice!). Sections s of L are identified with functions $\dot{s}$ on M by means of the non-vanishing global section s_o :

$$s_o(m) = (m,1) \quad \wedge \quad s(m) = \dot{s}(m)s_o(m) = (m, \dot{s}(m)),$$

the connection on L is given by

$$(\nabla_\zeta s)^\cdot = \zeta\dot{s} - \frac{i}{\hbar}\theta(\zeta)\dot{s}\ , \qquad \theta = \frac{1}{2}(p\,dq - q\,dp)$$

and the compatible inner product by

$$(s_1,s_2)(m) = \overline{\dot{s}_1(m)} \cdot \dot{s}_2(m)$$

<u>1B. The polarization P_{hol} and the bundle $B^{P_{hol}}$</u>: $P_{hol} = \mathbb{C} \cdot X_{p+iq} \Rightarrow$ $\Rightarrow P \cap \bar{P} = \{0\}$ so $D = \{0\}$ and $E = T^*\mathbb{R}^2$. This implies that $M/D \cong M$ and P_{hol} is an admissible Kähler polarization. Because we can identify M with M/D, the foliation $\tilde{E}$ can be identified with E and we see that $\tilde{E}$ has only one leaf: M and on it a complex structure is defined by

$$J \operatorname{Re} v = \operatorname{Im} v, \quad v \in P$$

$$\Longleftrightarrow J\frac{\partial}{\partial p} = \frac{\partial}{\partial q}, \quad J\frac{\partial}{\partial q} = -\frac{\partial}{\partial p}.$$

This is the standard complex structure on $\mathbb{R}^2 \cong \mathbb{C}$ with complex coordinate $z = p+iq$, hence if we speak (in the sequel of this example) of a holomorphic function on $M \cong M/D$, we mean a function holomorphic in the complex coordinate $z = p+iq$.

Since D is globally spanned by hamiltonian vector fields (a trivial remark because $D = \{0\}$) there exists a global non-vanishing section ν_o of $B^{P_{hol}}$ which is covariant constant along P; it is defined by:

$$\nu_o(m, X_{p+iq}) = |\omega(X_{p+iq}, \bar{X}_{p+iq})|^{-\frac{1}{4}} = 2^{-\frac{1}{4}}.$$

By means of this $-\frac{1}{2}$-P-density ν_o each section ν of $B^{P_{hol}}$ defines a unique function $\dot{\nu}$ on M by

$$\nu = \dot{\nu} \cdot \nu_o$$

and since ν_o is covariant constant along P we have:

$$(\nabla_\zeta \nu)^\cdot = \zeta\dot{\nu}, \quad \zeta \in E^{\mathbb{C}}.$$

<u>1C. Quantization:</u> since there exist global non-vanishing sections s_o and ν_o of L and $B^{P_{hol}}$, we have a global trivializing section $\psi_o = s_o \otimes \nu_o$

of QB. Each section ψ of QB defines a unique function $\dot{\psi}$ on M by:

$$\psi = \dot{\psi} \cdot \psi_o$$

and we have for $\zeta \in E^{\mathbb{C}}$:

$$(\nabla_\zeta \psi)^\cdot = \zeta\dot{\psi} - \frac{i}{\hbar}\theta(\zeta)\dot{\psi} \quad , \qquad \theta = \tfrac{1}{2}(p\,dq - q\,dp).$$

If ψ_1 and ψ_2 are covariant constant along P, then they define a density on M/D = M by:

$$(\psi_1,\psi_2)(m, X_{p+iq}, X_{p-iq}) = \overline{\dot{\psi}_1}(m)\dot{\psi}_2(M) \cdot \overline{\nu_o(X_{p+iq})} \cdot \nu_o(X_{p+iq}) \cdot$$

$$\cdot\, |\omega(X_{p+iq}, X_{p-iq})| \cdot |\omega(X_{p+iq}, X_{p-iq})|^{\frac{1}{2}}$$

$$\Longleftrightarrow \quad (\psi_1,\psi_2)(m, X_{p+iq}, X_{p-iq}) = 2\,\overline{\dot{\psi}_1(m)}\dot{\psi}_2(m)$$

$$\Longleftrightarrow \quad (\psi_1,\psi_2)(m, \frac{\partial}{\partial p}, \frac{\partial}{\partial q}) = \overline{\dot{\psi}_1(m)} \cdot \dot{\psi}_2(m).$$

We know from the theory that if we can find one global non-vanishing section ψ_c which is covariant constant along P, then all other sections of QB which are covariant constant along P are given by functions g on M/D = M which are holomorphic with respect to the complex structure on the leaves of E, i.e. by functions g(p,q) which satisfy the Cauchy-Riemann equations

$$\left(\frac{\partial}{\partial p} + i\frac{\partial}{\partial q}\right) g(p,q) = 0 \iff X_{p+iq}\, g = 0.$$

the condition ψ covariant constant along P is given by:

$$X_{p+iq}\,\dot{\psi} - \frac{i}{\hbar}\,\theta(X_{p+iq})\dot{\psi} = 0$$

$$\Longleftrightarrow \quad \left(\frac{\partial}{\partial p} + i\frac{\partial}{\partial q}\right)\dot{\psi} = \frac{i}{2\hbar}(ip-q)\dot{\psi} = -\frac{1}{2\hbar}(p+iq)\dot{\psi}$$

and a solution $\dot{\psi}_c$ is given by:

$$\dot{\psi}_c(p,q) = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(-\frac{p^2+q^2}{4\hbar}\right).$$

This ψ_c is globally non-vanishing so if ψ is any section of QB which is covariant constant along P then $\psi = g \cdot \psi_c$ where g is holomorphic, hence we can identify the (pre-)Hilbert space with the space of holomorphic functions with a special inner product:

$$H = \{g: \mathbb{C} \to \mathbb{C} \mid g \text{ holomorphic and } \frac{1}{2\pi\hbar} \iint |g(p+iq)|^2 e^{-\frac{p^2+q^2}{2\hbar}} dpdq < \infty\}$$

$$< g_1, g_2 > = \frac{1}{2\pi\hbar} \iint \overline{g_1(p+iq)}\, g_2(p+iq)\, e^{-\frac{p^2+q^2}{2\hbar}} dpdq.$$

The condition on an observable f to be quantizable is:

$$\left[\frac{\partial f}{\partial p}\frac{\partial}{\partial q} - \frac{\partial}{\partial q}\frac{\partial}{\partial p}, \frac{\partial}{\partial p} + i\frac{\partial}{\partial q}\right] = \alpha\left(\frac{\partial}{\partial p} + i\frac{\partial}{\partial q}\right), \quad \alpha \in \mathbb{C}$$

$$\Longleftrightarrow \frac{\partial^2 f}{\partial p^2} = \frac{\partial^2 f}{\partial q^2} \wedge \frac{\partial^2 f}{\partial p \partial q} = 0$$

$$\Longleftrightarrow f(p,q) = a \cdot \left(\frac{p^2+q^2}{2}\right) + bp + cq + d, \qquad a,b,c,d \in \mathbb{R}.$$

Consequently, we can quantize the observables $H = \frac{1}{2}(p^2+q^2)$, p and q. We start to compute $\delta(H)$ and therefore we have to know $L_{X_H} \nu_o$. The flow ρ_t associated to X_H is given by:

$$\rho_t(q,p) = (q\cos t + p\sin t, \ -q\sin t + p\cos t)$$

$$\rho_{t*}\left(\frac{\partial}{\partial p} + i\frac{\partial}{\partial q}\right) = e^{-it}\left(\frac{\partial}{\partial p} + i\frac{\partial}{\partial q}\right)$$

$$\Longleftrightarrow (L_{X_H}\nu_o)(m, X_{p+iq}) = \frac{d}{dt}\Big|_{t=0} \nu_o(\rho_t m, e^{-it} X_{p+iq}) =$$

$$= \frac{d}{dt}\Big|_{t=0}\left[\nu_o(\rho_t m, X_{p+iq})|e^{-it}|^{-\frac{1}{2}}\right]$$

$$= \frac{d}{dt}\Big|_{t=0}(2^{-\frac{1}{4}}) = 0 \ .$$

With these ingredients we can compute $\delta(H)$:

$$\delta(H)\psi = \delta(H)(\dot{\psi}\, s_o \otimes \nu_o) = \delta(H)(g \cdot \dot{\psi}_c \cdot s_o \otimes \nu_o)$$

$$= -i\hbar(\nabla_{X_H} g\psi_c\, s_o) \otimes \nu_o - i\hbar g\dot{\psi}_c\, s_o \otimes L_{X_H}\nu_o + Hg\dot{\psi}_c\, s_o \otimes \nu_o$$

$$= \hbar(p+iq)g'(p+iq)\dot{\psi}_c\, s_o \otimes \nu_o$$

where $g'(z)$ denotes $\frac{dg}{dz}$; this result can be stated in terms of our interpretation of the Hilbert space as:

$$\delta(H)g(z) = \hbar\, zg'(z).$$

The eigenfunctions of the hamilton operator $\delta(H)$ are given by

$$\delta(H)g = Eg \iff zg'(z) = \frac{E}{\hbar}\, g(z).$$

Since $g(z)$ is a holomorphic function on $\mathbb{C}$ it follows that $g(z)$ should be a homogeneous polynomial of degree $E/\hbar$ hence:

$$E_n = n\hbar\,, \quad g_n(z) = z^n, \quad n \in \mathbb{N}\,.$$

We can also compute the operators $\delta(p)$ and $\delta(q)$:

$$L_{X_p}\nu_o = L_{X_q}\nu_o = 0$$

"hence": $\delta(p)g(z) = \frac{1}{2}zg(z) + \hbar g'(z)$

$$\delta(q)g(z) = -\frac{i}{2}zg(z) + i\hbar g'(z).$$

Since $H = \frac{1}{2}(p^2+q^2)$, it is interesting to compute the operator $\frac{1}{2}(\delta(p)^2+\delta(q)^2)$:

$$\tfrac{1}{2}[\delta(p)^2 + \delta(q)^2]g(z) = \hbar z g'(z) + \tfrac{1}{2}\hbar g(z)$$

which has the same eigenfunctions as $\delta(H)$ but it has different eigenvalues:

$$\tfrac{1}{2}(\delta(p)^2 + \delta(q)^2)g_n(z) = (n+\tfrac{1}{2})\hbar g_n(z).$$

The fact that $\delta(H)$ and $\tfrac{1}{2}(\delta(p)^2 + \delta(q)^2)$ have different eigenvalues should not come as a big surprise because we nowhere required of the representation δ that it should satisfy $\delta(f_1 \cdot f_2) = \delta(f_1) \cdot \delta(f_2)$. The fact that $\delta(H)$ has not the correct eigenvalues is due to the use of the absolute values in the definition of a $-\tfrac{1}{2}$-P-density (as already noted in section 7 and a solution will be given in section 11).

<u>1D. Summary</u>: $M = T^*\mathbb{R}$, $L = M \times \mathbb{C}$, $P = P_{hol}$ "imply":

$$H = \{g: \mathbb{C} \to \mathbb{C} \mid \iint |g|^2 e^{-\frac{|z|^2}{2\hbar}} dpdq < \infty\},$$

$f: M \to \mathbb{R}$ is quantizable iff $f = aH + bp + cq + d$, $a, b, c, d \in \mathbb{R}$.

$$\delta(H) = \hbar z \frac{d}{dz}, \quad \delta(p) = \tfrac{1}{2}z + \hbar \frac{d}{dz}, \quad \delta(q) = -\frac{i}{2}z + i\hbar \frac{d}{dz}$$

$$\tfrac{1}{2}(\delta(p)^2 + \delta(q)^2) = \hbar\left(\tfrac{1}{2} + z\frac{d}{dz}\right).$$

<u>EXAMPLE 2</u>: *the moving free particle on* S^2: *energy representation.*
In this example we want to quantize the symplectic manifold $M = T^*S^2 \setminus \{0\}$, i.e. the cotangent bundle of S^2 without the zero section, together with the canonical symplectic form $\omega = d\theta$, $\theta = p_i dq^i$. In order to visualize M we will embed M into $\mathbb{R}^6$ as the submanifold of $\mathbb{R}^6$ defined by:

$$M = \{(\vec{x},\vec{y}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |\vec{x}| = 1, \ \vec{x}\cdot\vec{y} = 0, \ |\vec{y}| \neq 0\}$$

This embedding looks like the tangent bundle of S^2, but that is to be expected: we have embedded S^2 into $\mathbb{R}^3$ and the natural metric on S^2 is derived from the natural metric on $\mathbb{R}^3$. Each metric on a manifold X defines an identification between TX and T^*X so $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ can be viewed either as $T\mathbb{R}^3$ or as $T^*\mathbb{R}^3$ with inner product/pairing:

$$T_{\vec{x}}\mathbb{R}^3 \ni \xi = v^i \frac{\partial}{\partial x^i} \simeq (\vec{x},\vec{v}), \qquad T^*_{\vec{x}}\mathbb{R}^3 \ni \alpha = w_i dx^i \simeq (\vec{x},\vec{w})$$

$$\Rightarrow \qquad \alpha(\xi) = \vec{v} \cdot \vec{w}$$

and it is exactly this identification which is inherited by S^2 as embedded submanifold of $\mathbb{R}^3$.

On $\mathbb{R}^6 = T^*\mathbb{R}^3$ we have a canonical 1-form $\theta = y_i dx^i$ and it is an elementary calculation to show that when we restrict θ to M (i.e. in fact the pullback of θ to M via the embedding) then we get the canonical 1-form θ of $M = T^*S^2 \setminus \{0\}$. This implies that all calculations concerning the symplectic form on M can be performed by using the symplectic form in $\mathbb{R}^6$, but ... calculation of the hamilton vector field associated to a function on M can give troubles! It would be nice if f is a function on M, $\tilde{f}$ a function on $\mathbb{R}^6$ such that $\tilde{f}|_M = f$, that in that case $X_f = X_{\tilde{f}}|_M$ but the extension $\tilde{f}$ is not unique and the different choices of $\tilde{f}$ give different $X_{\tilde{f}}$ (on $\mathbb{R}^6$ even when restricted to M). However, one can show that if $X_{\tilde{f}}$ is tangent to M (as a submanifold of $\mathbb{R}^6$) then $X_{\tilde{f}}|_M = X_f$.

After this rather lenghty introduction of M we proceed with the observables we want to quantize; these are in the first place the kinetic energy of a free particle on S^2 given by

$$H = \tfrac{1}{2}|\vec{y}|^2$$

which is just the quadratic form on T^*S^2 defined by the natural metric on S^2, and in the second place the three components of the angular momentum $\vec{L}$:

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix} \iff \vec{L} = \vec{x} \wedge \vec{y}$$

where $\wedge$ denotes the cross product or vector product of two vectors in $\mathbb{R}^3$; these four observables are not independent since the relation

$$\tfrac{1}{2}(L_1^2 + L_2^2 + L_3^2) = H$$

holds. If we want to compute the associated hamilton vector fields, we are at once confronted with the technical difficulties mentioned above: $X_{\frac{1}{2}|\vec{y}|^2}$ is not tangent to M! The conditions $|\vec{x}| = 1$ and $\vec{x} \cdot \vec{y} = 0$ imply that a tangent vector $\xi \in T_{(\vec{x},\vec{y})}\mathbb{R}^6$ given by

$$\xi = \vec{v} \cdot \frac{\partial}{\partial \vec{x}} + \vec{w} \cdot \frac{\partial}{\partial \vec{y}}$$

is tangent to M iff $\vec{v} \cdot \vec{x} = 0$ *and* $\vec{w} \cdot \vec{x} + \vec{v} \cdot \vec{y} = 0$.
With these results it is easy to show that the hamiltonian vector field (on $\mathbb{R}^6$) of the function $H = \frac{1}{2}|\vec{y}|^2 \cdot |\vec{x}|^2$ (which coincides with the previous definition of H on M!) is indeed tangent to M hence

$$X_H = \vec{y} \cdot \frac{\partial}{\partial \vec{x}} - |\vec{y}|^2 \vec{x} \cdot \frac{\partial}{\partial \vec{y}} \qquad (\text{on } M).$$

The hamiltonian vector fields of L_i pose no problems:

$$X_{L_1} = x^2 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^2} + y_2 \frac{\partial}{\partial y_3} - y_3 \frac{\partial}{\partial y_2}$$

$$X_{L_2} = x^3 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^3} + y^3 \frac{\partial}{\partial y^1} - y^1 \frac{\partial}{\partial y^3}$$

$$X_{L_3} = x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1} + y_1 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial y_1} .$$

Since we want to quantize these observables, we have to choose the polarization P such that they belong to the class of quantizable observables.

The polarization P_{en} we will use is due to K.Ii (see [Ii]) and one of its main features is that it contains X_H as the only real direction (which implies that H is automatically quantizable). As a consequence the real distribution D associated to P_{en} is spanned by X_H and then we are interested whether M/D is a manifold or not and if so, if $\pi: M \to M/D$ is a submersion.

The flow ρ_t associated to X_H is given by:

$$\rho_t(\vec{x},\vec{y}) = (\vec{x}\cos|\vec{y}|t + \frac{\vec{y}}{|\vec{y}|}\sin|\vec{y}|t\ ,\ -\vec{x}|\vec{y}|\sin|\vec{y}|t + \vec{y}\cos|\vec{y}|t)$$

so the leaves of D are circles $(\rho_{2\pi/|\vec{y}|}(\vec{x},\vec{y}) = (\vec{x},\vec{y}))$.

We want to argue that $M/D \cong S^2 \times \mathbb{R}^+$ where the projection π is given by:

$$\pi(\vec{x},\vec{y}) = (\vec{n},v) = (\vec{x} \wedge \frac{\vec{y}}{|\vec{y}|},\ |\vec{y}|).$$

Therefore observe that a leaf of D (= an integral curve of X_H) represents a geodesic on S^2 (the first three coordinates) together with the velocity of rotation (the second three coordinates). Each geodesic of S^2 determines a plane in $\mathbb{R}^3$ (the plane which contains the geodesic) and hence two points on S^2 (the intersection of S^2 with the normal to this plane). If we now realize that the velocity with which this geodesic is passed cannot be zero (we have omitted the zero section of T^*S^2 !) we can use the sign of this velocity to choose one of those two intersections. It is exactly this what is done: the vector $\vec{x} \wedge \vec{y}$ is normal to the plane which determines the geodesic (i.e. the integral curve of X_H through $(\vec{x},\vec{y}) \in M$) and $\frac{\vec{x} \wedge \vec{y}}{|\vec{y}|}$ is one of the two intersections of this normal with S^2. The only missing information concerning this geodesic (if we have the point on S^2) is the absolute value of the travelling velocity which is given by $|\vec{y}|$. This explanation should give some meaning to the formula we gave for the canonical projection $\pi: M \to M/D$. It should be noted that it is essential that we have omitted the zero section of T^*S^2 because the formula of π is not defined for $|\vec{y}| = 0$ and moreover X_H vanishes when $|\vec{y}| = 0$ so the dimension of D would not be constant on T^*S^2 (one should compare this situation with the harmonic oscillator in the energy representation of section 7).

Since the projection $\pi: M \to M/D$ will play an important role, we will give local charts U_a $(a = 1,2,3)$ on M on which the projection becomes trivial: $U_a \subset S^2 \times \mathbb{R}^+ \times S^1$ and $\pi : U_a \to S^2 \times \mathbb{R}^+$ is just the projection on the first two factors. On U_a we use coordinates $(\vec{n}, v, t_a)$ where $\vec{n} \in S^2$, $v \in \mathbb{R}^+$ and $\exp(i\,t_a) \in S^1$: $t_a \in \mathbb{R}$ is a cyclic coordinate (if we are careful we have to divide U_a into two charts in order to cover S^1 by two patches (e.g. $(0,2\pi)$ and $(-\pi,\pi)$), but we hope the reader will allow us the use of this formally incorrect method in order to simplify the reasoning). We first define the vectors e_a as $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ and with these we define the vectors $\vec{x}_a(\vec{n})$ by

$$\vec{x}_a(\vec{n}) = \frac{\vec{e}_a \wedge \vec{n}}{|\vec{e}_a \wedge \vec{n}|} \qquad (|\vec{e}_a \wedge \vec{n}|^2 = 1 - n_a^2).$$

With these ingredients we define the local charts $\phi_a : U_a \to M$ by

$$U_a = \{(\vec{n}, v, t_a) \mid 1 - n_a^2 \neq 0\}$$

$$\phi_a(\vec{n}, v, t_a) = (\vec{x}_a(\vec{n}) \cos t_a + \vec{n} \wedge \vec{x}_a(n) \sin t_a, \quad \vec{y} = \vec{n} \wedge \vec{x} \cdot v)$$

hence by definition it follows that $\pi \circ \phi_a(\vec{n}, v, t_a) = (\vec{n}, v) \in S^2 \times \mathbb{R}^+$. The transition functions between the charts U_a are given by:

$$\phi_b^{-1} \circ \phi_a : (\vec{n}, v, t_a) \longmapsto (\vec{n}, v, t_b = t_a + \alpha_{ba}(\vec{n}))$$

where $\alpha_{ba}(\vec{n})$ is defined by

$$\exp(i\,\alpha_{ba}(\vec{n})) = \vec{x}_a(\vec{n}) \cdot (\vec{x}_b(\vec{n}) + i\,\vec{n}\wedge\vec{x}_b(\vec{n}));$$

in particular $\alpha_{ab} = -\alpha_{ba}$, $\alpha_{aa} = 0$ and

$$\exp(i\,\alpha_{21}(\vec{n})) = -\frac{n_1 n_2 + i\,n_3}{\sqrt{(1-n_1^2)(1-n_2^2)}} .$$

The inverse mapping ϕ_a^{-1} is given by:

$$\phi_a^{-1} : (\vec{x},\vec{y}) \longmapsto (\vec{n}, v, t_a) = \left(\frac{\vec{x}\wedge\vec{y}}{|\vec{y}|}, |\vec{y}|, t_a\right)$$

$$\exp(i\,t_a) = \vec{x}\cdot(\vec{x}_a(n) + i\,\vec{n}\wedge\vec{x}_a(\vec{n})).$$

With these calculation we finish the introduction of this example and we go on with the quantization.

2A. Prequantization: as usual (it becomes a habit) we take $L = M \times \mathbb{C}$ with global non-vanishing section $s_o(m) = (m,1)$ and we identify sections s of L with functions $\dot{s}$ on M by $s(m) = (m, \dot{s}(m))$. In terms of these identifications the connection ∇ becomes $(\nabla_\zeta s)^{\cdot} = \zeta\dot{s} - \frac{i}{\hbar}\theta(\zeta)\dot{s}$ and the inner product becomes $(s_1,s_2)(m) = \overline{\dot{s}_1(m)}\cdot\dot{s}_2(m)$.

2B. The polarization P_{en} and the bundle $B^{P_{en}}$: the polarization P_{en} is defined by

$$P_{en}\Big|_{(\vec{x},\vec{y})\in M} = \mathbb{C}\cdot X_H + \mathbb{C}\,\vec{n}\cdot\left(\frac{\partial}{\partial\vec{x}} - iv\frac{\partial}{\partial\vec{y}}\right)$$

where $\vec{n} = \dfrac{\vec{x}\wedge\vec{y}}{|\vec{y}|}$, $v = |\vec{y}|$. If we write $V_R = \vec{n}\cdot\dfrac{\partial}{\partial\vec{x}}$, $V_I = \vec{n}v\cdot\dfrac{\partial}{\partial\vec{y}}$ and $V = V_R - i\,V_I$, then:

$$[X_H,V] = iV \quad , \quad [V_R,V_I] = \frac{1}{v}X_H .$$

Because $D = \mathbb{R}\cdot X_H$, $E = \mathbb{R}X_H \oplus \mathbb{R}V_R \oplus \mathbb{R}V_I$ it follows from the above commutation relations that P_{en} and E are involutive and that D has constant dimension 1, so, because $\omega(X_H,V) = 0$, P_{en} satisfies all requirements of a complex polarization according to a proposition of section 8; moreover, since M/D is a manifold and $\pi: M \to M/D$ a submersion, P_{en} is admissible. At this point it is rather difficult to find a complex function w such that $V = \lambda X_w + \mu X_H$, $\lambda, \mu \in \mathbb{C}$, but later on we will see how to obtain such a function; it will turn out that the (local) function w on $\mathbb{R}^6$ defined by

$$w = \frac{x_3y_1 - x_1y_3 + i(x_1y_2 - x_2y_1)}{\frac{1}{2}|\vec{y}|^2\cdot(|\vec{x}|^2+1) - x_2y_3 + x_3y_2}$$

has a hamiltonian vector field (on $\mathbb{R}^6$) tangent to M and

$$X_w = \lambda\cdot V \quad \text{(on the domain of } w, \text{ for some function } \lambda).$$

In order to compute the complex structure defined on the leaves of $\tilde{E} = \pi_* E$ in M/D, we first compute the vector fields X_H and V in terms of the coordinate charts U_a:

on U_a: $\quad X_H = v\dfrac{\partial}{\partial t_a}$

$(n_a^2 \neq 1)$ $\quad V = i\left(\dfrac{\vec{y}}{v} + i\vec{x}\right)\cdot\dfrac{\partial}{\partial\vec{n}} + \dfrac{n_a}{1-n_a^2}\left(\dfrac{y_a}{v} + ix_a\right)\dfrac{\partial}{\partial t_a}$

where $\vec{x}$ and $\vec{y}$ are defined by the map ϕ_a^{-1}.
We see that the leaves of the foliation $\tilde{E}$ are precisely the balls $S^2\times\{v\}$ $(v\in\mathbb{R}^+)$ and the complex structure on $S^2\times\{v\}$ defined by P_{en} turns S^2 into $\mathbb{P}^1(\mathbb{C})$:

$$J\left(\vec{w}\cdot\frac{\partial}{\partial\vec{n}}\right) = \vec{w}'\cdot\frac{\partial}{\partial\vec{n}}$$

where $\vec{w}'$ is obtained by rotating $\vec{w}$ over $90°$ in the plane perpendicular to $\vec{n}$ such that the triple $(\vec{w}', \vec{w}, \vec{n})$ has the same orientation as $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$. This characterization of the complex structure on S^2 is the same as the statement that the function w_a defined on $\pi(U_a)$ by

$$w_1 = \frac{n_2+in_3}{1-n_1}, \quad w_2 = \frac{n_3+in_1}{1-n_2}, \quad w_3 = \frac{n_1+in_2}{1-n_3}$$

is a (local) holomorphic coordinate on $S^2 \times \{v\} \cong \mathbb{P}^1(\mathbb{C})$.
According to the theory the associated hamiltonian vector fields are vector fields in P_{en} and (again) we have to modify these functions (seen as functions on $\mathbb{R}^6$) such that their hamilton vector fields are tangent to M, which forces us to write $\frac{1}{2}(|\vec{x}|^2+1) - n_a$ instead of $1 - n_a$ in the denominator of w_a. Furthermore, if we express the function w_b in terms of the complex coordinate w_a on $S^2 \cong \mathbb{P}^1(\mathbb{C})$ then w_b ought to be a holomorphic function and indeed:

$$w_1 = \frac{w_2+i}{w_2-i}, \quad w_2 = i\frac{w_1+1}{w_1-1}, \quad w_3 = \frac{w_i+i}{w_1-i} .$$

We now turn our attention to the bundle $B^{P_{en}}$: D is globally spanned by the hamiltonian vector field X_H hence there exists a global non-vanishing section ν_o of $B^{P_{en}}$ which is covariant constant along P ; on the local chart U_a it is defined by:

$$\nu_o(X_H, V) = |\varepsilon_{\omega,1}(V,\bar{V})|^{-\frac{1}{4}} = |\omega(V,\bar{V})|^{-\frac{1}{4}} = |-2iv|^{-\frac{1}{4}} .$$

Using this global trivialization, each section ν of $B^{P_{en}}$ can be identified with a function $\dot{\nu}$ on M by $\nu = \dot{\nu} \cdot \nu_o$ and the connection (along $E^{\mathbb{C}}$) is given by:

$$(\nabla_\zeta \nu)^{\cdot} = \zeta\dot{\nu} \qquad (\zeta \in E^{\mathbb{C}}) .$$

<u>2C. Quantization</u>: the global sections s_o of L and ν_o of $B^{P_{en}}$ define a global trivializing section $\psi_o = s_o \otimes \nu_o$ of QB with which we can identify sections ψ of QB with functions $\dot\psi$ on M by $\psi = \dot\psi\psi_o$; the connection on QB is given by:

$$(\nabla_\zeta \psi)^{\cdot} = \zeta\dot\psi - \frac{i}{\hbar}\theta(\zeta)\dot\psi .$$

If ψ_1 and ψ_2 are covariant constant along P then they define a density on $S^2 \times \mathbb{R}^+$ (with coordinates $(\vec{n},v)$) by:

$$\begin{aligned}(\psi_1,\psi_2)\left(\pi_*\left(V,\bar V,\frac{\partial}{\partial v}\right)\right) &= \overline{\dot\psi_1(m)}\dot\psi_2(m)\cdot\overline{\nu_o(X_H,V)}\cdot\nu_o(X_H,V)\cdot \\ &\quad\cdot\left|\varepsilon_\omega\left(X_H,V,\bar V,\frac{\partial}{\partial v}\right)\right|\cdot\left|\varepsilon_{\omega,1}(V,\bar V)\right|^{\frac12} \\ &= \overline{\dot\psi_1(m)}\dot\psi_2(m)\cdot\left|\varepsilon_\omega\left(X_H,V,\bar V,\frac{\partial}{\partial v}\right)\right| .\end{aligned}$$

On S^2 there exists a natural density called $d\Omega$ associated to the metric on S^2 and defined by:

$$d\Omega\left(\vec w\cdot\frac{\partial}{\partial\vec n}, w'\cdot\frac{\partial}{\partial\vec n}\right) = |\vec w\wedge\vec w'|$$

(N.B. this density is usually denoted in polar coordinates on S^2 by $d\Omega = \sin\theta\, d\theta\, d\phi$). Using this density we obtain:

$$(\psi_1,\psi_2)\left(\vec w\cdot\frac{\partial}{\partial\vec n},\vec w'\cdot\frac{\partial}{\partial\vec n},\frac{\partial}{\partial v}\right) = \overline{\dot\psi_1(m)}\dot\psi_2(m)v^2\, d\Omega(\vec w,\vec w')$$

$$\Rightarrow \langle\psi_1,\psi_2\rangle = \int_{M/D}(\psi_1,\psi_2) = \int_{\mathbb{R}^+}dv\int_{S^2}d\Omega\,\overline{\dot\psi_1(m)}\dot\psi_2(m)v^2 .$$

The next step in the identification of the Hilbert space H is the determination of a local/global section ψ which is covariant constant along P. On the chart U_a the condition to be covariant constant is given by:

$$\begin{cases} X_H\dot{\psi}_a - \frac{i}{\hbar}\theta(X_H)\dot{\psi}_a = 0 \\ V\dot{\psi}_a - \frac{i}{\hbar}\theta(V)\dot{\psi}_a = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} v\frac{\partial\dot{\psi}_a}{\partial t_a} = \frac{i}{\hbar}v^2\dot{\psi}_a(\vec{n}, v, t_a) \\ i\left(\frac{\vec{y}}{v} + i\vec{x}\right)\frac{\partial\psi_a}{\partial\vec{n}} + \frac{n_a}{1-n_a^2}\left(\frac{y_a}{v} + ix_a\right)\frac{\partial\psi_a}{\partial t_a} = 0 \end{cases}$$

and it is easy to show that a solution is given by:

$$\dot{\psi}_a(n, v, t_a) = h_a(v) \cdot (1-n_a^2)^{v/2\hbar} \cdot \exp(ivt_a/\hbar).$$

However, we now have the same troubles as with the harmonic oscillator in the energy representation: the cyclic nature of t_a implies that $h_a(v) = 0$ whenever $v/\hbar \notin \mathbb{Z}$. Since it is easy to show (using the equation $\frac{\partial\dot{\psi}}{\partial t_a} = i(v/\hbar)\dot{\psi}$ and the cyclic nature of t_a) that all sections ψ which are covariant constant along P should be zero whenever $v/\hbar \notin \mathbb{Z}$, it follows that H reduces to $\{0\}$ since then $<\psi_1, \psi_2> = 0$ always ($\mathbb{Z}^+$ has measure zero in $(\mathbb{R}^+, \lambda^{(1)})$). However, we will ignore this point and we will assume (as in the case of the harmonic oscillator) that the integration over v has to be replaced by a summation over all allowed discrete values of v (which should be the correct answer if we had things like distribution valued sections). We will see that if we adopt this point of view, then things turn out very nice.

To remind ourselves that v should be restricted to a discrete set of values, we replace v by the variable $\lambda = v/\hbar$ which then takes values in $\mathbb{Z}^+$. We now choose $h_a(\lambda) = 1/\hbar\lambda$ so

$$\dot{\psi}_a(\vec{n}, \lambda, t_a) = \frac{1}{\hbar\lambda}\sqrt{1 - n_a^2}^{\,\lambda} \exp(i\lambda t_a)$$

and: $$<\psi_a,\psi_a> = \sum_{\lambda=1}^{\infty} \int d\Omega \ (1-n_a^2)^{\lambda}$$

which explains our choice of $h_a(\lambda)$: to get rid of the factor v^2 in the inner product.

It should be noted that in the view we just have adopted, ψ_a is a *non-vanishing* section of QB over U_a (although it is zero if $v \notin \mathbb{Z}^+$). The question which now arises is: can we extend ψ_a to a global section of QB while it remains covariant constant along P? Since $U_1 \cup U_2 = M$ we calculate the correspondence between ψ_1 and ψ_2 on $U_1 \cap U_2$ using the complex coordinate w_1 (resp. w_2) instead of the coordinates $\vec{n}$:

$$\text{on } U_1 \cap U_2: \quad \psi_2 = \left(\frac{1-w_1^2}{2w_1}\right)^{\lambda} \psi_1 \iff \psi_1 = \left(i\,\frac{1+w_2^2}{2w_2}\right)^{\lambda} \psi_2 \,.$$

These formula show that the global section ψ of QB defined by:

$$\psi\big|_{U_1} = \psi_1 \,, \qquad \psi\big|_{U_2} = \left(i\,\frac{1+w_2^2}{2w_2}\right)^{\lambda} \psi_2$$

is covariant constant along P because it differs (locally) by a holomorphic function from a covariant constant one. Moreover: ψ is the section ψ_1 which is extended by zero outside U_1 (the points $w_2 = \pm i$ just cover $M \setminus U_1$).

Now suppose ψ is any section of QB which is covariant constant along P, then (according to the theory) there exist functions $g_1(\lambda,w_1)$ on U_1 and $g_2(\lambda,w_2)$ on U_2 which are holomorphic with respect to the coordinate w such that:

$$\psi\big|_{U_1} = g_1(\lambda,w_1)\psi_1 \,, \qquad \psi\big|_{U_2} = g_2(\lambda,w_2)\psi_2$$

hence on the intersection $U_1 \cap U_2$ we have:

$$g_2(\lambda,w_2)\psi_2 = g_1(\lambda,w_1)\psi_1 = g_1\left(\lambda,\frac{w_2+i}{w_2-i}\right)\left(i\,\frac{1+w_2^2}{2w_2}\right)^{\lambda}\psi_2$$

$$\Longleftrightarrow\quad g_2(\lambda,w_2) = g_1\left(\lambda,\frac{w_2+i}{w_2-i}\right)\left(\frac{(w_2+i)(w_2-i)}{-2iw_2}\right)^{\lambda} .$$

Since $g_1(\lambda,w)$ and $g_2(\lambda,w)$ are defined on $\mathbb{Z}^+\times(\mathbb{C}\backslash\{0\})$, the function $g_1(\lambda,w)$ has a Laurent series

$$g_1(\lambda,w) = \sum_{m=-\infty}^{\infty} a_{\lambda m}\, w^m \quad , \quad a_{\lambda m}\in\mathbb{C}$$

and since $g_2(\lambda,w)$ has no pole in $w_2 = \pm i$, it follows that this series is finite:

$$g_1(\lambda,w) = \sum_{m=-\lambda}^{\lambda} a_{\lambda m}\, w^m .$$

If we now realize that U_1 is dense in M, then we see that the function $g_1(\lambda,w_1)$ determines the function $g_2(\lambda,w_2)$ completely; g_1 and g_2 together determine the section ψ so each section ψ of QB which is covariant constant along P is characterized by a function $g(\lambda,w)$ of the form:

$$g(\lambda,w) = \sum_{m=-\lambda}^{\lambda} a_{\lambda m}\, w^m \quad , \quad a_{\lambda m}\in\mathbb{C}$$

$$\psi = g(\lambda,w_1)\psi_1 .$$

The inner product between two of these sections is given by:

$$\psi = g(\lambda,w_1)\psi_1 \quad , \quad g(\lambda,w) = \sum_{m=-\lambda}^{\lambda} a_{\lambda m}\, w^m$$

$$\hat{\psi} = \hat{g}(\lambda,w_1)\psi_1 \quad , \quad \hat{g}(\lambda,w) = \sum_{m=-\lambda}^{\lambda} \hat{a}_{\lambda m}\, w^m$$

$$<\psi,\hat{\psi}> = \sum_{\lambda=1}^{\infty}\int_{S^2} d\Omega \sum_{m,m'=-\lambda}^{\lambda} \bar{a}_{\lambda m}\,\hat{a}_{\lambda m'} \; , \; \bar{w}_1^m\, w_1^{m'}(1-n_1^2)^{\lambda}$$

$$= \sum_{\lambda=1}^{\infty}\int_{S^2} d\Omega \sum_{m,m'} \bar{a}_{\lambda m}\,\hat{a}_{\lambda m'}\left(\frac{n_2-in_3}{1-n_1}\right)^{m}\left(\frac{n_2+in_3}{1-n_1}\right)^{m'}(1-n_1^2)^{\lambda} =$$

$$= \sum_{\lambda=1}^{\infty}\int_{\mathbb{R}^2}\frac{4dpdq}{(r^2+1)^2}\sum_{m,m'} a_{\lambda m}\,\hat{a}_{\lambda m'}(p-iq)^m(p+iq)^{m'}\left(\frac{2r}{r^2+1}\right)^{2\lambda}$$

$$\boxed{r^2 = p^2+q^2}$$

and the Hilbert space H is given by

$$H = \{\psi: M \to QB \mid \psi = \left(\sum_{m=-\lambda}^{\lambda} a_{\lambda m} w_1^m\right)\psi_1, \ <\psi,\psi> < \infty\}$$

N.B. The sections $\psi \in H$ differ from the *global* section ψ_1 by a function $g(\lambda,w_1)$ which is holomorphic in w_1 ; however, ψ_1 has zero's so g may have poles (whose order should be less than or equal to the order of the zero of ψ_1) .

After the determination of the Hilbert space H we now turn our attention to the observables H and L_i (i =1,2,3). Because $X_H \in P_{en}$ the associated operator $\delta(H)$ is just multiplication by $H = \frac{1}{2}v^2 = \frac{1}{2}\hbar^2\lambda^2$ so:

$$\delta(H)\psi = \tfrac{1}{2}v^2\psi = \tfrac{1}{2}\hbar^2\lambda^2\psi$$

The eigenvalues of $\delta(H)$ are $E_\ell = \frac{1}{2}\hbar^2\ell^2$ with corresponding eigenfunctions $\psi_{\ell m}$ $(|m| \leqslant \ell)$ defined by:

$$\psi_{\ell m} = \delta_{\ell\lambda}\, w_1^m\, \psi_1$$

(i.e. the coefficients $a_{\lambda m}$, are all zero except $a_{\ell m}$ which equals 1) and we see that the eigenspace of E_ℓ has dimension $2\ell + 1$.

To compute the operators $\delta(L_i)$ we observe that

$$[X_H, X_{L_i}] = [V, X_{L_i}] = 0 \ , \quad X_{L_i} v = 0$$

so if ρ_t is the flow associated to X_{L_i} then:

$$\rho_{t*}X_H = X_H \ , \quad \rho_{t*}V = V$$

hence:

$$\left(L_{X_{L_i}} \nu_o\right)(m, X_H, V) = \frac{d}{dt}\Big|_{t=0} \nu_o\left(X_H|_{\rho_t m}, V|_{\rho_t m}\right)$$

$$= \frac{d}{dt}\Big|_{t=0} \left(\underbrace{| -2iv|^{-\frac{1}{4}}}_{\text{at } \rho_t m}\right) = X_{L_i} | -2iv|^{-\frac{1}{4}} = 0 \quad (X_{L_i} v=0)$$

and finally, since $\theta(X_{L_i}) = L_i$ (L_i is linear in the momentum y_j) it follows that:

$$\delta(L_i)\dot{\psi}\, s_o \otimes \nu_o = -i\hbar(X_{L_i}\dot{\psi})s_o \otimes \nu_o \,.$$

Because we have expressed each $\psi \in H$ as a product of a function $g(\lambda,w_1)$ on U_1 and ψ_1, we "have to" know the vector fields X_{L_i} in terms of the coordinates on U_i:

$$X_{L_1} = (\vec{e}_1\wedge\vec{n})\cdot\frac{\partial}{\partial\vec{n}}$$

$$X_{L_2} = (\vec{e}_2\wedge\vec{n})\cdot\frac{\partial}{\partial\vec{n}} + \frac{n_2}{1-n_1^2}\cdot\frac{\partial}{\partial t_1}$$

$$X_{L_3} = (\vec{e}_3\wedge\vec{n})\cdot\frac{\partial}{\partial\vec{n}} + \frac{n_3}{1-n_1^2}\cdot\frac{\partial}{\partial t_1}$$

hence on U_1 we have:

$$X_{L_1}w_1 = iw_1 \,, \quad X_{L_2}w_1 = -\frac{i}{2}(w_1^2-1) \,, \quad X_{L_3}w_1 = -\frac{w_1^2+1}{2}$$

$$X_{L_1}\dot{\psi}_1 = 0 \,, \quad X_{L_2}\dot{\psi}_1 = \frac{i}{2}\cdot\frac{w_1^2+1}{w_1}\dot{\psi}_1 \,, \quad X_{L_3}\psi_1 = \frac{\lambda}{2}\cdot\frac{w_1^2-1}{w_1}\dot{\psi}_1$$

and finally the action of $\delta(L_i)$ on $\psi_{\ell m}$:

$$\delta(L_1)\psi_{\ell m} = m\hbar\,\psi_{\ell m}$$

$$\delta(L_2)\psi_{\ell m} = \frac{\hbar}{2}\Big((\ell-m)\psi_{\ell,m+1} + (\ell+m)\psi_{\ell,m-1}\Big)$$

$$\delta(L_3)\psi_{\ell m} = -i\frac{\hbar}{2}\Big((\ell-m)\psi_{\ell,m+1} - (\ell+m)\psi_{\ell,m-1}\Big) \,.$$

To conclude our computations we mention that

$$\tfrac{1}{2}\Big(\delta(L_1)^2 + \delta(L_2)^2 + \delta(L_3)^2\Big)\psi_{\ell m} = \tfrac{1}{2}\hbar^2\ell(\ell+1)\psi_{\ell m} \,.$$

2D. Summary and conclusions: $M = T^{*}S^{2} \setminus \{0\}$, $L = M \times \mathbb{C}$ and $P = P_{en}$ imply (if we forget the problems concerning the need of distribution valued sections):

$$H = \{\psi: M \to QB \mid \psi\left(\sum_{m=-\lambda}^{\lambda} a_{\lambda m} w_1^m\right)\psi_1 \wedge \langle\psi,\psi\rangle < \infty\}$$

$$\langle\psi,\hat{\psi}\rangle = \sum_{\lambda=1}^{\infty} \int_{S^2} d\Omega \sum_{m,m'=-\lambda}^{\lambda} \bar{a}_{\lambda m}\, \hat{a}_{\lambda m'}\, \bar{w}_1^m \cdot w_1^{m'} \,(1-n_1^2)^{\lambda}.$$

The observables $H = \frac{1}{2}|\vec{y}|^2 = \frac{1}{2}\hbar^2\lambda^2$ and $\vec{L} = \vec{x} \wedge \vec{y}$ are quantizable; the eigenvalues E_ℓ of $\delta(H)$ are given by $E_\ell = \frac{1}{2}\hbar^2\ell^2$ with corresponding eigenfunctions $\psi_{\ell m}$ $(|m| \leq \ell)$

$$\psi_{\ell m} = \delta_{\ell\lambda}\, w_1^m\, \psi_1 \,, \qquad \delta(H)\psi_{\ell m} = E_\ell\, \psi_\ell = \tfrac{1}{2}\hbar^2\ell^2\, \psi_{\ell m}.$$

The operator $\frac{1}{2}\left(\delta(L_1)^2 + \delta(L_2)^2 + \delta(L_3)^2\right)$ has the same eigenfunctions $\psi_{\ell m}$ as $\delta(H)$, but different eigenvalues:

$$\tfrac{1}{2}\left(\delta(L_1)^2 + \delta(L_2)^2 + \delta(L_3)^2\right)\psi_{\ell m} = \tfrac{1}{2}\hbar^2\ell(\ell+1)\psi_{\ell m}.$$

As in the case of the harmonic oscillator in the holomorphic representation (the previous example) this result should not come as a total surprise. However, we have seen that the quantization procedure as described in section 9 is not the final quantization procedure because it does not give the correct quantization of the energy of the harmonic oscillator (energy levels $E_n = n\hbar$ instead of $E_n = (n+\frac{1}{2})\hbar$). To correct this (small) error we need the metalinear correction and when we have it, we have to investigate its effect upon the energy eigenvalues of this example!

We leave it to the reader to philosophize about the resemblance between the $\psi_{\ell m} \in H$ as eigenfunctions of the operator $\delta(L_1)^2 + \delta(L_2)^2 + \delta(L_3)^2$ and the eigenfunctions $Y_{\ell m} \in L_2(S^2, d\Omega)$ of the Laplace-Beltrami operator Δ on S^2 $\left(\Delta = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$.

11 THE METALINEAR CORRECTION

As we have said in earlier sections, the errors in the energy eigenvalues of the harmonic oscillator can be traced back to the use of the absolute value in the definition of a $-\frac{1}{2}$-P-density. Let us recall the definitions: R is the bundle of all P-frames, i.e. the fibre R_m consists of all bases $(\xi_1,\ldots,\xi_n)$ of $P_m \subset T_m M^{\mathbb{C}}$: R is a principal $GL(n,\mathbb{C})$ bundle "since" all such frames can be indexed by $GL(n,\mathbb{C})$. A $-\frac{1}{2}$-P-density ν is a function $\nu\colon R \to \mathbb{C}$ such that

$$\forall m \in M \quad \forall(\xi_1,\ldots,\xi_n) \in R_m \quad \forall g \in GL(n,\mathbb{C})$$

$$\nu(m,\ (\xi_1,\ldots,\xi_n)\cdot g) = \nu(m,\ (\xi_1,\ldots,\xi_n)) \cdot |\det g|^{-\frac{1}{2}}.$$

If we wish to get rid of the absolute value of $\det g$, we come into troubles with the definition of the square root: which branch of the square root do we need? Consider the matrix

$$g(t) = \exp(2\pi i t/n)\cdot \mathrm{id}(\mathbb{C}^n) \in GL(n,\mathbb{C}), \qquad t \in [0,1];$$

this is a continuous function of t but, if we do wish the function

$$\nu(m,\ (\xi)\cdot g(t)) = \nu(m,\ (\xi)) \quad \det g(t)^{-\frac{1}{2}}$$

to be continuous, we always have troubles at $t = 0$ and $t = 1$, in whatever way we choose the square root of $\det g$.

One way to solve this problem is to add the information concerning the square root to the matrix g itself. In this way one obtains the metalinear group $ML(n,\mathbb{C})$ which consists of pairs $(g,z) \in GL(n,\mathbb{C}) \times (\mathbb{C}\setminus\{0\})$ satisfying:

$$z^2 = \det g$$

with multiplication $(g_1,z_1)\cdot(g_2,z_2) = (g_1g_2,z_1z_2)$ which is a correct group action on $ML(n,\mathbb{C})$. Another way of visualizing the group $ML(n,\mathbb{C})$ is as a subgroup of $GL(n+1,\mathbb{C})$:

$$(g,z) \cong \begin{pmatrix} g & \vdots & \emptyset \\ \cdots & \vdots & \cdots \\ \emptyset & \vdots & z \end{pmatrix} \in GL(n+1,\mathbb{C}).$$

Associated with $ML(n,\mathbb{C})$ are two group homomorphisms: p and λ:

$$p: ML(n,\mathbb{C}) \longrightarrow GL(n,\mathbb{C}) \quad : (g,z) \longmapsto g$$

$$\lambda: ML(n,\mathbb{C}) \longrightarrow \mathbb{C}^* = \mathbb{C}\backslash\{0\} : (g,z) \longmapsto z .$$

The map p is a nontrivial 2-1 covering of $GL(n,\mathbb{C})$: there exist local continuous sections $s: GL(n,\mathbb{C}) \to ML(n,\mathbb{C})$, but no global ones.

One now might think that we have solved our problems: there is a natural action of $ML(n,\mathbb{C})$ on R:

$$(\xi_1,\ldots,\xi_n) \cdot (g,z) \stackrel{\text{def}}{=} (\xi_1,\ldots,\xi_n) \cdot g$$

and we define a $-\frac{1}{2}$-P-density ν as a function ν on R satisfying:

$$\nu((\xi) \cdot (g,z)) = \nu((\xi)) \cdot z^{-1} \qquad (z^2 = \det g)$$

$$\Longleftrightarrow \nu((\xi) \cdot \tilde{g}) = \nu((\xi)) \cdot \lambda(\tilde{g})^{-1}, \quad \tilde{g} \in ML(n,\mathbb{C})$$

but, ... this is ridiculous because $(\xi) \cdot (id,-1) = (\xi)$ hence $\nu(\xi) = -\nu(\xi)$ implying that ν should be identically zero. From this observation it is obvious that we have to incorporate the metalinear group not only in the action on R, but in the fibres of R themselves: we need to change the principal $GL(n,\mathbb{C})$ bundle R into a principal $ML(n,\mathbb{C})$ bundle $\tilde{R}$, just by replacing the fibres $R_m \cong GL(n,\mathbb{C})$ by $\tilde{R}_m \cong ML(n,\mathbb{C})$. We want furthermore that $\tilde{R}$ resembles R as much as possible: whenever there is a twist in R there should be a twist in $\tilde{R}$, but how do we formulate this wish correctly?

Suppose $\{U_\alpha, g_{\alpha\beta}\}$ is a trivializing cover of R with transition functions $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL(n,\mathbb{C})$. Suppose furthermore that there exist (continuous!) lifts $\tilde{g}_{\alpha\beta}\colon U_\alpha \cap U_\beta \to ML(n,\mathbb{C})$ such that $p \circ \tilde{g}_{\alpha\beta} = g_{\alpha\beta}$ and satisfying the cocycle condition $\tilde{g}_{\alpha\beta} \cdot \tilde{g}_{\beta\gamma} = \tilde{g}_{\alpha\gamma}$, *then* $\{U_\alpha, \tilde{g}_{\alpha\beta}\}$ defines a principal $ML(n,\mathbb{C})$ bundle $\tilde{R}$ over M which resembles R in the way required. However, it is not evident that such $\tilde{g}_{\alpha\beta}$ always exist; it is always possible to choose the U_α such that the lifts $\tilde{g}_{\alpha\beta}$ exist, but it is not necessary that these lifts satisfy the cocycle condition. In fact, whether this is possible depends upon the cohomology class in $H^2(M, \mathbb{Z}/2\mathbb{Z})$ determined by R: if it vanishes then such lifts satisfying the cocycle condition do exist. Moreover, the different possible choices (if there are any) are characterized by $H^1(M, \mathbb{Z}/2\mathbb{Z})$. (see [Simms & Woodhouse])

To summarize these results: a principal $ML(n,\mathbb{C})$ bundle $\tilde{R}$ over M with the desired properties does exist if the cohomology class in $H^2(M, \mathbb{Z}/2\mathbb{Z})$ determined by R vanishes; the different possibilities are characterized by $H^1(M, \mathbb{Z}/2\mathbb{Z})$. Associated to such a bundle $\tilde{R}$ (and in the sequel we will always assume that $\tilde{R}$ exists) is a projection $\tilde{p}\colon \tilde{R} \to R$ which commutes with the bundle projections:

$$\begin{array}{ccc} \tilde{R} & \xrightarrow{\tilde{p}} & R \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & M & \end{array}$$

and which is defined in a local chart U_α by:

$$\tilde{p}(m, (g,z)) = (m,g)$$

$$\Longleftrightarrow \quad \tilde{p}(m,\tilde{g}) = (m, p(\tilde{g})), \qquad \tilde{g} \in ML(n,\mathbb{C}) .$$

When we identify R as the bundle of P-frames then

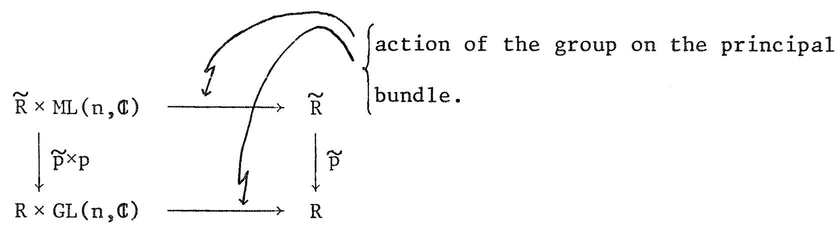
$$\tilde{p}(m,\tilde{g}) = (m,f_m)$$

where $f_m = (\xi_1,\ldots,\xi_n)$ is a frame (basis) of $P_m \subset T_m M^{\mathbb{C}}$ determined by the matrix $p(\tilde{g})$. Since the inverse image $\tilde{p}^{-1}(m,f_m)$ consists of two points (the two square roots of $\det g$ if $g \sim f_m$), the bundle $\tilde{R}$ is also called the bundle of metaframes. It follows from the construction of $\tilde{R}$ that if $f\colon U \subset M \to R$ is a local section of R ($f(m)$ is a frame at m, i.e. $f(m) = (\xi_1|_m,\ldots,\xi_n|_m)$ where $\xi_1,\ldots,\xi_n$ are vector fields on U which span P), then there exists (at least locally in U) a lift $\tilde{f}\colon U' \subset U \to \tilde{R}$ such that $\tilde{p} \circ \tilde{f} = f$. This observation, together with some obvious considerations, shows that the map $\tilde{p}\colon \tilde{R} \to R$ is a 2-1 covering of $\tilde{R}$ over R.

To state our results in a different way: $\tilde{R}$ is a principal $ML(n,\mathbb{C})$ bundle over M and $\tilde{p}\colon \tilde{R} \to R$ is a 2-1 covering which is simultaneously a map of M-bundles. Moreover, if $\tilde{f} \in \tilde{R}_m$ is a metaframe and $\tilde{g} \in ML(n,\mathbb{C})$ then $\tilde{f}\cdot\tilde{g} \in \tilde{R}_m$ and $\tilde{p}$ satisfies the relation

$$\tilde{p}(\tilde{f}\cdot\tilde{g}) = \tilde{p}(\tilde{f}) \cdot p(\tilde{g}) \;, \qquad p\colon ML(n,\mathbb{C}) \to GL(n,\mathbb{C})$$

(this is a consequence of the construction of $\tilde{R}$: $p(\tilde{g}_{\alpha\beta}) = g_{\alpha\beta}$), a relation which is often stated as the commutativity of the following diagram:



Having defined a bundle $\tilde{R}$ (if it exists) we now can define a $-\frac{1}{2}$-P-form as the generalization of a $-\frac{1}{2}$-P-density which does not use the absolute value: a $-\frac{1}{2}$-P-form $\tilde{\nu}$ is a function $\tilde{\nu}\colon \tilde{R} \to \mathbb{C}$ satisfying

$$\forall \tilde{g} = (g,z) \in ML(n,\mathbb{C}) \quad \forall m \in M \quad \forall \tilde{f}_m \simeq \tilde{R}_m \cong ML(n,\mathbb{C})$$

$$\tilde{\nu}(m, \tilde{f}_m \cdot (g,z)) = \tilde{\nu}(m, \tilde{f}_m) \cdot z^{-1} \qquad (z^2 = \det g)$$

or $$\tilde{\nu}(m, \tilde{f}_m \cdot \tilde{g}) = \tilde{\nu}(m, \tilde{f}_m)\lambda(\tilde{g})^{-1} .$$

In the same way as before we define the line-bundle $\tilde{B}^P$ over M as the bundle whose fibre $\tilde{B}^P_m$ consists of all functions $\tilde{\nu}_m : \tilde{R}_m \to \mathbb{C}$ satisfying:

$$\tilde{\nu}_m(\tilde{f}_m \cdot \tilde{g}) = \tilde{\nu}_m(\tilde{f}_m) \cdot \lambda(\tilde{g})^{-1}$$

which is a line-bundle since such a function $\tilde{\nu}_m$ is determined completely by its value on a fixed metaframe $\tilde{f}_o$. It then follows that sections $\tilde{\nu}$ of $\tilde{B}^P$ can be identified (in the obvious way) with $-\frac{1}{2}$-P-forms as defined before. We can describe the line-bundle $\tilde{B}^P$ using the same trivializing cover as we used to define the bundle $\tilde{R}$: $\tilde{B}^P$ has local charts $U_\alpha \times \mathbb{C}$ with transition functions $\mu_{\alpha\beta} : U_\beta \times \mathbb{C} \to U_\alpha \times \mathbb{C}$

$$\mu_{\alpha\beta} : (m,z) \longmapsto (m, \mu_{\alpha\beta}(m) \cdot z), \quad \mu_{\alpha\beta}(m) \in \mathbb{C}^*$$

where $\mu_{\alpha\beta}(m)$ is defined by:

$$\mu_{\alpha\beta}(m) = \lambda(\tilde{g}_{\alpha\beta}(m))$$

because: if $\tilde{\nu}$ is a $-\frac{1}{2}$-P-form on M, then it defines local $-\frac{1}{2}$-P-forms $\tilde{\nu}_\alpha$ on the local charts $U_\alpha \times ML(n,\mathbb{C})$ of $\tilde{R}$ and on the intersection of U_α and U_β we have:

$$\tilde{\nu}_\beta(m, \tilde{id}) = \tilde{\nu}_\alpha(m, \tilde{g}_{\alpha\beta}(m) \cdot \tilde{id}) = \tilde{\nu}_\alpha(m, \tilde{id})\lambda(\tilde{g}_{\alpha\beta}(m))^{-1}$$

so our claim follows if we identify $\tilde{\nu}$ with the set of local functions $\tilde{\nu}_\alpha(m, \tilde{id})$ on U_α.

At this point it is interesting to note that $\tilde{B}^P$ need not be a trivial bundle; the fact that the bundles B^P over M and $\Delta^r X$ over X (X an arbitrary manifold) are trivial depends strongly upon the use of absolute values! The difference between $\Lambda^n X$ and $\Delta^1 X$ (volume forms and densities on X) is just the absolute value: $\Lambda^n X$ can be nontrivial ($\Longleftrightarrow X$ is not orientable), while $\Delta^1 X$ is always trivial. The distinction between $\tilde{B}^P$ and B^P is of the same nature although the difference is slightly more than just the absolute value. It is this analogy which explains the name -½-P-form: forms (i.e. k-forms on a manifold X) do not use the absolute value, densities do.

12 QUANTIZATION III

In this section we will trace the consequences of replacing B^P by $\tilde{B}^P$. We have to start with the definition of a (partial) connection on $\tilde{B}^P$ analogous to the connection ∇ on B^P. This connection ∇ on $\tilde{B}^P$ will be defined for vector in P only, contrary to B^P where it was defined for vectors in $E^{\mathbb{C}} = P + \bar{P}$. Suppose $\zeta \in P$ and $\tilde{\nu}$ a $-\frac{1}{2}$-P-form (i.e. $\tilde{\nu}$ is a section of $\tilde{B}^P$) then we define a new $-\frac{1}{2}$-P-form $\nabla_\zeta \tilde{\nu}$ in the following way. Let $m_o \in M$, let $\tilde{f}_o \in \tilde{R}_{m_o}$ be a metaframe at m_o and let $(\xi_1,\ldots,\xi_n) = \tilde{p}(\tilde{f}_o) \in R_{m_o}$ be the associated P-frame at m_o. Then there exist hamiltonian vector fields $\eta_1, \ldots, \eta_n$ defined in a neighbourhood U of m_o which span P on U (by definition of a complex polarization) such that $\eta_i|_{m_o} = \xi_i$; since $\tilde{p}$ is a 2-1 covering there exists a neighbourhood U' of m_o, $U' \subset U$ and a local section $\tilde{f}: U' \to \tilde{R}$ of $\tilde{R}$ satisfying the conditions:

$$\tilde{f}(m_o) = \tilde{f}_o \quad , \qquad \tilde{p} \circ \tilde{f} = f = (\eta_1,\ldots,\eta_n) \; ;$$

we will call such a section $\tilde{f}$ a local hamiltonian metaframe on U' (what we have shown is that for any $\tilde{f}_o \in \tilde{R}_{m_o}$ there exists a neighbourhood U' of m_o and a local hamiltonian metaframe $\tilde{f}$ on U' such that $\tilde{f}(m_o) = \tilde{f}_o$). The $-\frac{1}{2}$-P-form $\nabla_\zeta \tilde{\nu}$ is now defined in the point $(m_o, \tilde{f}_o) \in R$ by

$$(\nabla_\zeta \tilde{\nu})(m_o, \tilde{f}_o) = \zeta|_{m_o} \tilde{\nu}(m, \tilde{f}(m)) .$$

<u>PROPOSITION</u>: $\nabla_\zeta \tilde{\nu}$ *is a well-defined* $-\frac{1}{2}$-P-*form*.

<u>PROOF</u>: let $\tilde{f}'$ be another local hamiltonian metaframe on U' around m_o, then there exists a function $\tilde{g}: U' \to ML(n,\mathbb{C})$ such that

$$\tilde{f}'(m) = \tilde{f}(m) \cdot \tilde{g}(m)$$

(remember: $\widetilde{R}$ is a principal $ML(n,\mathbb{C})$ bundle). Define the hamiltonian vector fields $\eta'_1, \ldots, \eta'_n$ on U' by $(\eta'_1|_m, \ldots, \eta'_n|_m) = \widetilde{p}(\widetilde{f}(m))$ and define the function $g: U' \to GL(n,\mathbb{C})$ by $g(m) = p(\widetilde{g}(m))$ then:

$$\eta'_i|_m = \eta_j|_m \, g_{ji}(m).$$

Because η_i and η'_j are hamiltonian vector fields in P it follows that g_{ji} is constant along P, in particular we have $\zeta|_{m_o} g_{ji}(m) = 0$ (the proof is analogous to the similar proofs in sections 6,9). We now observe that $\lambda(\widetilde{g}(m)) = \sqrt{\det g(m)}$ for some branch of the square root in the neighbourhood U', hence

$$\zeta|_{m_o} \lambda(\widetilde{g}(m)) = 0.$$

Finally: $(\nabla_\zeta \widetilde{\nu})(m_o, \widetilde{f}'(m_o)) = \zeta|_{m_o} \widetilde{\nu}(m, \widetilde{f}'(m)) = \zeta|_{m_o} \widetilde{\nu}(m, \widetilde{f}(m)\widetilde{g}(m))$

$$= \zeta|_{m_o} [\widetilde{\nu}(m, \widetilde{f}(m))\lambda(\widetilde{g}(m))^{-1}] = [\zeta|_{m_o} \widetilde{\nu}(m, \widetilde{f}(m))]\lambda(\widetilde{g}(m_o))^{-1}$$

$$= (\nabla_\zeta \widetilde{\nu})(m_o, \widetilde{f}(m_o))\lambda(\widetilde{g}(m_o))^{-1}. \quad \boxed{\text{QED}}$$

<u>PROPOSITION</u>: ∇ *is a (partial) flat connection on* B^P.

<u>PROOF</u>: we only prove that ∇ is flat, i.e. if $\xi, \zeta \in P$ then

$$\nabla_\zeta \nabla_\xi \widetilde{\nu} - \nabla_\xi \nabla_\zeta \widetilde{\nu} = \nabla_{[\zeta\ \xi]} \widetilde{\nu}$$

the other properties of a connection follow easily from the definitions. Let $\widetilde{f}(m)$ be a local hamiltonian metaframe around $m_o \in M$ then:

$$(\nabla_\xi \widetilde{\nu})(m, \widetilde{f}(m)) = \xi|_m \widetilde{\nu}(m', \widetilde{f}(m'))$$

hence: $(\nabla_\zeta \nabla_\xi \widetilde{\nu})(m_o, \widetilde{f}(m_o)) - (\nabla_\xi \nabla_\zeta \widetilde{\nu})(m_o, \widetilde{f}(m_o))$

$$= \zeta|_{m_o} \xi|_m \widetilde{\nu}(m', \widetilde{f}(m')) - \xi|_{m_o} \zeta|_m \widetilde{\nu}(m', \widetilde{f}(m'))$$

$$= [\zeta,\xi]|_{m_o} \widetilde{\nu}(m, \widetilde{f}(m)) = (\nabla_{[\zeta,\xi]} \widetilde{\nu})(m_o, \widetilde{f}(m_o)). \quad \boxed{\text{QED}}$$

With these proposition we have finished the construction of the bundle $\tilde{B}^P$ with its partial flat connection. The next step is to define the bundle QB (abuse of notation since it is a bundle different from the bundle QB defined in section 9):

$$QB := L \otimes \tilde{B}^P$$

where L is the prequantization line-bundle. On QB is defined a partial connection ∇ in the obvious way: if $\zeta \in P$, $\psi = s \otimes \tilde{\nu}$ a section of QB then:

$$\nabla_\zeta \psi = (\nabla_\zeta s) \otimes \tilde{\nu} + s \otimes (\nabla_\zeta \tilde{\nu}).$$

On the sections ψ of QB which are covariant constant along P we want to define an inner product in the same way as before; suppose $\psi_i = s_i \otimes \tilde{\nu}_i$, $i = 1,2$ are two sections of QB, $m \in M$ and suppose $\tilde{f} \in \tilde{R}_m$. Define $(\zeta_1,\dots,\zeta_n) = \tilde{p}(\tilde{f})$ and choose $(\xi_1,\dots,\xi_n) \in T_m M^{\mathbb{C}}$ in such a way that the following conditions are satisfied:

$$(\zeta_1,\dots,\zeta_n,\ \xi_1,\dots,\xi_n) \text{ is a basis of } T_m M^{\mathbb{C}}$$

$$(\zeta_1,\dots,\zeta_k) \text{ is a basis of } D_m^{\mathbb{C}} = P_m \cap \bar{P}_m.$$

The first condition is a condition on $(\xi_1,\dots,\xi_n)$, the second condition restricts the possible choices of $\tilde{f}$ (remember that $(\zeta_1,\dots,\zeta_n)$ is a basis of P_m). With these ingredients we define a function $(\psi_1,\psi_2)_m$ on $F^{2n-k}_{\pi(m)}(M/D)^{\mathbb{C}}$ by:

$$(\psi_1,\psi_2)_m(\pi_*(\zeta_{k+1},\dots,\zeta_k,\ \xi_1,\dots,\xi_n)) = (s_1,s_2)(m)\overline{\tilde{\nu}_1(m,\tilde{f})} \cdot$$

$$\cdot\ \tilde{\nu}_2(m,\tilde{f}) \cdot |\varepsilon_{\omega,k}(\zeta_{k+1},\dots,\zeta_n,\ \bar{\zeta}_{k+1},\dots,\bar{\zeta}_n)|^{\frac{1}{2}} \cdot$$

$$\cdot\ |\varepsilon_\omega(\zeta_1,\dots,\zeta_n,\ \xi_1,\dots,\xi_n)|.$$

As before, this function is not defined on the whole of $F^{2n-k}_{\pi(m)}(M/D)^{\mathbb{C}}$ but one can prove as in section 9 that it defines a 1-density:

PROPOSITION: $(\psi_1,\psi_2)_m$ *defines a unique* 1*-density on* $F^{2n-k}_{\pi(m)}(M/D)^{\mathbb{C}}$.

Furthermore, by a simple adaption of the proof in case of $-\frac{1}{2}$-P-densities (just using the lemma of section 8 completely) one can prove the following proposition:

PROPOSITION: *if* ψ_1 *and* ψ_2 *are covariant constant along* P *then* $(\psi_1,\psi_2)_m$ *defines a unique density* (ψ_1,ψ_2) *on* M/D, *i.e.* $(\psi_1,\psi_2)_m$ *depends only upon* $\pi(m)$.

REMARK: as in the case of $-\frac{1}{2}$-P-densities one can construct local $-\frac{1}{2}$-P-forms $\tilde{\nu}$ which are covariant constant along P and non-vanishing: let $\tilde{f}(m)$ be a local hamiltonian metaframe, then the $-\frac{1}{2}$-P-form $\tilde{\nu}_o$ (defined where $\tilde{f}$ is defined) given by

$$\tilde{\nu}_o(m, \tilde{f}(m)) = 1$$

is covariant constant along P and non-vanishing as promised. Hence if ψ is a local section of QB then there exists a (local!) section s of L such that $\psi = s \otimes \nu_o$ and moreover: ψ is covariant constant along P iff s is covariant constant along P.

As before we define a prehilbert space PH by

$$PH = \{\psi: M \to QB \mid \forall \zeta \in P : \nabla_\zeta \psi = 0 \wedge \int_{M/D} (\psi,\psi) < \infty\}$$

with the "inner product":

$$<\psi_1,\psi_2> = \int_{M/D} (\psi_1,\psi_2)$$

and we take the associated Hilbert space H as the Hilbert space of the quantum mechanical description of the classical system described by the symplectic manifold (M,ω). From the definitions as given above, one can (easily) deduce that, even if one uses $-\frac{1}{2}$-P-forms instead of $-\frac{1}{2}$-P-densities, if ψ_1 and ψ_2 are two local sections of QB which are both covariant constant along P, then they differ by a function on M/D which is holomorphic when restricted to a leaf of $\tilde{E} = \pi_* E$ (which had and has a complex structure induced by P) .

The last step in the quantization procedure is to define the quantizable observables: a function $f: M \to \mathbb{R}$ is quantizable iff $[X_f,P] \subset P$; this is the same condition as before, but now the associated quantum operator has to be defined in the new situation. Suppose ζ is a real vector field on M preserving P , i.e. $[\zeta,P] \subset P$, then if ρ_t denotes the associated flow and if $\xi \in P_m$ then $\rho_{t*}\xi \in P_{\rho_t m}$; it follows that ρ_t induces an action on P-frames: $f = (\xi_1,\ldots,\xi_n) \in R_m \Rightarrow (\rho_{t*}\xi_1,\ldots,\rho_{t*}\xi_n) \in R_{\rho_t m}$. Now suppose $\tilde{f} \in \tilde{R}_m$ then (if t is small enough) there exists a unique (continuous) action called $\tilde{\rho}_{t*}$ on $\tilde{f}$ such that

$$\tilde{\rho}_{t*}\tilde{f} \in \tilde{R}_{\rho_t m} \quad \text{and} \quad \tilde{p}(\tilde{\rho}_{t*}\tilde{f}) = \rho_{t*}(\tilde{p}\tilde{f})$$

(to see it, we note that $\tilde{f}$ determines a branch of the square root (of the frame $f = \tilde{p}\tilde{f} \in R_m$) which can be extended in a neighbourhood of m) . Using this action we can define the operator L_ζ on a $-\frac{1}{2}$-P-form $\tilde{\nu}$ by:

$$(L_\zeta\tilde{\nu})(m,\tilde{f}) = \frac{d}{dt}\Big|_{t=0} \tilde{\nu}(\rho_t m, \tilde{\rho}_{t*}\tilde{f})$$

Another way to define the operator L_ζ is the following: associated to the flow ρ_{t*} on R is a vector field on R which we will call $\hat{\zeta}$:

$$\hat{\zeta}_{(m,f)} \text{ "=" } \frac{d}{dt}\Big|_{t=0} \rho_{t*}(m)f$$

and $\hat{\zeta}$ obviously satisfies the relation $\pi_*\hat{\zeta} = \zeta$ where $\pi: R \to M$ is the bundle projection.

Associated to the flow $\tilde{\rho}_{t*}$ on $\tilde{R}$ is a vector field $\tilde{\zeta}$ on $\tilde{R}$ which satisfies the relations:

$$\tilde{p}_*\tilde{\zeta} = \hat{\zeta} \quad , \quad \pi_*\tilde{\zeta} = \zeta \qquad (\text{here } \pi: \tilde{R} \to M) \ .$$

Since $\tilde{p}$ is a covering, especially since it is locally a diffeomorphism, we also can define $\tilde{\zeta}$ as the unique lift of $\hat{\zeta}$ to $\tilde{R}$ such that $\tilde{p}_*\tilde{\xi} = \hat{\zeta}$ and with these definitions (especially of $\tilde{\zeta}$ as the unique lift of $\hat{\zeta}$) the operator L_ζ can be defined as

$$L_\zeta\tilde{\nu} = \tilde{\zeta}\tilde{\nu} \ .$$

<u>PROPOSITION</u>: *$L_\zeta\tilde{\nu}$ is a well-defined $-\frac{1}{2}$-P-form and L_ζ possesses all properties of a Lie derivative (except that it is defined for a restricted class of vector fields):*

(i) $\qquad L_\zeta(\tilde{\nu}_1+\tilde{\nu}_2) = L_\zeta\tilde{\nu}_1 + L_\zeta\tilde{\nu}_2,\ L_\zeta(f\tilde{\nu}) = (\zeta f)\tilde{\nu} + f L_\zeta\tilde{\nu}, \quad f: M \to \mathbb{C}$

(ii) $\qquad L_{\zeta_1+\zeta_2}\tilde{\nu} = L_{\zeta_1}\tilde{\nu} + L_{\zeta_2}\tilde{\nu},\ \ L_{\alpha\zeta}\tilde{\nu},\ \ \alpha\ {}_{\zeta}\tilde{\nu},\ \ \alpha \in \mathbb{R}$

(iii) $\qquad L_{\zeta_1}L_{\zeta_2}\tilde{\nu} - L_{\zeta_2}L_{\zeta_1}\tilde{\nu} = L_{[\zeta_1,\zeta_2]}\tilde{\nu}$

Moreover, if $X_f \in P$ *then* $\nabla_{X_f}\tilde{\nu} = L_{X_f}\tilde{\nu} \qquad (f: M \to \mathbb{R})$

<u>PROOF</u>: we only prove the last statement, the others are left to the reader (condition (ii) and (iii) follow from the fact that $\zeta \longmapsto \hat{\zeta}$ is a Lie algebra homomorphism and the fact that $\tilde{p}$ is locally a diffeomorphism). Let ρ_t be the flow associated to X_f and let $f(m) = (\xi_1|_m,\ldots,\xi_n|_m)$ be a hamiltonian frame spanning P in a neighbourhood U of $m_o \in M$ with an associated hamiltonian metaframe $\tilde{f}(m) \in \tilde{R}_m$, $\tilde{p}\circ\tilde{f} = f$, then:

$$(\nabla_{X_f}\tilde{\nu})(m_o,\tilde{f}(m_o)) = X_f|_{m_o}\tilde{\nu}(m,\tilde{f}(m)) \ .$$

On the other hand, since X_f and ξ_i are hamiltonian vector fields in P it follows that $\rho_{t*}\xi_i = \xi_i$ hence

$$\rho_{t*}f(m_o) = f(\rho_t m_o)$$

and so $\tilde{\rho}_{t*}\tilde{f}(m_o) = \tilde{f}(\rho_t m_o)$ by definition.

Now: $$(L_{X_f}\tilde{\nu})(m_o, \tilde{f}(m_o)) = \frac{d}{dt}\Big|_{t=0} \tilde{\nu}(\rho_t m_o, \tilde{\rho}_{t*}\tilde{f}(m_o))$$
$$= X_f\big|_{m_o} \tilde{\nu}(m, \tilde{f}(m)) \; . \qquad \boxed{\text{QED}}$$

On the sections of QB we define an operator $\delta(f)$ for each quantizable observable f by the usual formula:

$$\delta(f)(s \otimes \tilde{\nu}) = (-i\hbar \nabla_{X_f} s + fs) \otimes \tilde{\nu} - i\hbar\, s \times L_{X_f}\tilde{\nu}$$

and we then have to prove that $\delta(f)$ maps the prehilbert space PH into PH and that $\delta(f)$ is essentially self-adjoint on H if X_f is complete. These proofs are the straightforward generalizations of the corresponding proofs in case of $-\frac{1}{2}$-P-densities, so we leave it to the reader to finish them.

This finishes the quantization procedure which includes the metalinear correction and we now turn our attention to the influence of the metalinear correction upon the examples given in previous sections.

13 SOME EXAMPLES III

In this section we apply the metalinear correction to some of the examples of previous sections and we will see its influence.

EXAMPLE 1: *an oriented configuration space with the vertical polarization.* In this example we quantize the symplectic manifold $M = T^*Q$ where Q is an oriented manifold, using the vertical polarization $P = D^{v\mathbb{C}}$. This example should be compared with example 3 of section 7; the condition "oriented" is sufficient to guarantee the existence of the bundle $\tilde{R}$.

1A. Prequantization: $L = M \times \mathbb{C} = T^*Q \times \mathbb{C}$ and we identify sections s of L with functions $\dot{s}$ on M in the usual way; the connection and compatible inner product become:

$$(\nabla_\zeta s)^\cdot = \zeta\dot{s} - \frac{i}{\hbar}\theta(\zeta)\dot{s}$$

$$(s_1,s_2)(m) = \overline{\dot{s}_1(m)}\,\dot{s}_2(m)$$

1B. The polarization $D^{v\mathbb{C}}$ and the bundles R, $\tilde{R}$ and $\tilde{B}^P$: the vertical polarization is spanned (locally) by the hamiltonian vector fields $X_{q^i} = -\frac{\partial}{\partial p_i}$ where q^j are the (local) coordinates on a coordinate chart U_α of Q. We now assume that $\{U_\alpha\}$ is a cover of Q of *oriented* coordinate charts, which implies that the determinant of the Jacobian of the transition functions is positive, i.e. if $\hat{q}^j$ are the coordinates on the local chart U_β then

$$\det\left(\frac{\partial\hat{q}^i}{\partial q^j}\right) \in \mathbb{R}^+ .$$

The local charts U_α of Q define a trivializing cover of R as follows: a chart U_α defines a chart $U_\alpha \times \mathbb{R}^n$ of $T^*Q = M$ and we then identify the element

$$((q,p),g) \in (U_\alpha \times \mathbb{R}^n) \times GL(n,\mathbb{C})$$

with the P-frame $((q,p),\ (X_{q^i})g_{ij})$.

If we denote by $J_{\beta\alpha}$ the Jacobian matrix of the map $\phi_{\beta\alpha}: U_\alpha \to U_\beta$ (the change of coordinates on Q), i.e.

$$(J_{\beta\alpha})_{ij} = \frac{\partial \hat{q}^i}{\partial q^j}$$

then, because $X_{\hat{q}^i} = \frac{\partial \hat{q}^i}{\partial q^j} X_{q^j}$, the correspondence between the local charts of R are given by

$$((\hat{q},\hat{p}),\ (X_{\hat{q}})g) \text{ "=" } ((q,p),\ (X_q)J^T_{\beta\alpha}g),$$

in other words, the transition functions $g_{\alpha\beta}(q,p)$ of the bundle R, $g_{\alpha\beta}: (U_\alpha \times \mathbb{R}^n) \cap (U_\beta \times \mathbb{R}^n) \to GL(n,\mathbb{C})$,

$$(U_\beta \times \mathbb{R}^n) \times GL(n,\mathbb{C}) \ni ((\hat{q},\hat{p}),g) \longmapsto ((q,p),\ g_{\alpha\beta}(q,p)\cdot g) \in U_\alpha \times \mathbb{R}^n \times GL$$

are given by:

$$g_{\alpha\beta}(q,p) = J^T_{\beta\alpha} = J^{-1T}_{\alpha\beta}$$

Since the determinant of the transition functions $g_{\alpha\beta}$ is real positive there exist lifts $\tilde{g}_{\alpha\beta}$ to $ML(n,\mathbb{C})$ defined by:

$$\tilde{g}_{\alpha\beta}(q,p) = (g_{\alpha\beta},\ +\sqrt{\det g_{\alpha\beta}}) \in ML(n,\mathbb{C})$$

such that the cocycle condition is satisfied. In conclusion we can say that the local charts U_α of an oriented atlas of Q define a trivializing cover of $\tilde{R}$ as a bundle over $M = T^*Q$; the projection $\tilde{p}: \tilde{R} \to R$ is given by

$$\tilde{p}((q,p),\tilde{g}) = ((q,p),\ (X_{q^1},\ldots,X_{q^n})\cdot p(\tilde{g}))$$

$$\Longleftrightarrow \tilde{p}((q,p),\ (g,z)) = ((q,p),\ (X_{q^i})g_{ij})$$

where we used the local charts $U_\alpha \times \mathbb{R}^n \times ML(n,\mathbb{C})$ and $U_\alpha \times \mathbb{R}^n \times GL(n,\mathbb{C})$.

According to the theory, the line-bundle $\tilde{B}^P$ has the same trivializing cover with transition functions $\mu_{\alpha\beta}$ defined by

$$\mu_{\alpha\beta}(q,p) = \lambda(\tilde{g}_{\alpha\beta}(q,p)) = +\sqrt{\det g_{\alpha\beta}(q)} \ .$$

It is interesting to note that in this case (and also in the other examples of this section) the bundle $\tilde{B}^P$ is trivial: let ρ_α be a locally finite partition of unity subordinated to the cover $\{U_\alpha \times \mathbb{R}^n\}$ of $M = T^*Q$ and define the local sections $\tilde{\nu}_\alpha : U_\alpha \times \mathbb{R}^n \to \tilde{B}^P$ of $\tilde{B}^P$ by

$$\tilde{\nu}_\alpha(q,p,\tilde{g}) = \lambda(\tilde{g})^{-1}$$

then the global section $\tilde{\nu} := \sum_\alpha \rho_\alpha \tilde{\nu}_\alpha$ of $\tilde{B}^P$ is non-vanishing, just because all the transition functions $\mu_{\alpha\beta}$ are real positive (N.B. we have chosen the positive square root of the positive numbers $\det g_{\alpha\beta}$).
If we remember that $\tilde{p}(q,p,\tilde{id}) = (q,p,(X_{q^i}))$, then we see that the local sections $\tilde{\nu}_\alpha$ defined above are covariant constant along P (the section $(q,p) \to (q,p,\tilde{id})$ is a local hamiltonian metaframe!). Now suppose $\tilde{\nu}$ is any $-\frac{1}{2}$-P-form (i.e. a section of $\tilde{B}^P$) then $\tilde{\nu}$ defines functions $\dot{\nu}^\alpha : U_\alpha \times \mathbb{R}^n \to \mathbb{C}$ by:

$$\tilde{\nu}\Big|_{U_\alpha \times \mathbb{R}^n} = \dot{\nu}^\alpha \cdot \tilde{\nu}_\alpha$$

and the connection ∇ on $\tilde{B}^P$ is given in terms of these local functions by:

$$(\nabla_\zeta \tilde{\nu})^{\cdot\alpha} = \zeta \dot{\nu}^\alpha$$

<u>1C. Quantization</u>: $QB = L \otimes \tilde{B}^P$ which has local sections $s_o \otimes \tilde{\nu}_\alpha$, where s_o is the global section of L defined by $s_o(m) = (m,1)$. For an arbitrary section $\psi = s \otimes \tilde{\nu}$ of QB we have:

$$\psi\Big|_{U_\alpha\times\mathbb{R}^n} = \dot{s}s_o \otimes \dot{\nu}^\alpha\cdot\tilde{\nu}_\alpha = \dot{\psi}^\alpha s_o \otimes \tilde{\nu}_\alpha$$

and with respect to these local sections $s_o \otimes \tilde{\nu}_\alpha$ the partial connection ∇ on QB becomes

$$(\nabla_\zeta\psi)^{\cdot\alpha} = \zeta\dot{\psi}^\alpha - \frac{i}{\hbar}\theta(\zeta)\dot{\psi}^\alpha$$

(remember that $\tilde{\nu}_\alpha$ is covariant constant along P), since this connection is defined for $\zeta \in P = D^{v\mathbb{C}}$ only and since $\theta = p_i\, dq^i$, we have $\theta(\zeta) = 0$ for $\zeta \in P$ hence

$$(\nabla_\zeta\psi)^{\cdot\alpha} = \zeta\dot{\psi}^\alpha .$$

In consequence, sections ψ of QB which are covariant constant along P can be identified (locally) with functions $\dot{\psi}^\alpha$ on $U_\alpha \subset Q$ (because $P \ni \zeta = \sum_i \zeta_i \frac{\partial}{\partial p_i}$). The relation between the different $\dot{\psi}^\alpha$ (considered as functions on $U_\alpha \subset Q$) is given by:

$$\begin{aligned}\dot{\psi}^\beta(\hat{q}) &= \dot{\psi}^\alpha(q)\cdot(\det g_{\alpha\beta}(q))^{\frac{1}{2}}\\ &= \dot{\psi}^\alpha(q)\cdot(\det J_{\alpha\beta}(q))^{-\frac{1}{2}}\end{aligned}$$

where $J_{\alpha\beta}$ is the Jacobian of the transition $\phi_{\alpha\beta}: U_\beta \to U_\alpha$, hence, apart from the square root, the $\dot{\psi}^\alpha$ behave as if they were volume forms on Q! When we used $-\frac{1}{2}$-P-densities we showed that the $\dot{\psi}^\alpha$'s behaved as $\frac{1}{2}$-densities on Q: a product of two such sections defines a density on Q; here a product of two such sections (without complex conjugation!) defines a volume form, which justifies the name $\frac{1}{2}$-forms on Q for these sections defined locally by the $\dot{\psi}^\alpha$'s.

However, since all Jacobians $J_{\alpha\beta}$ have a positive determinant, the $\dot{\psi}^\alpha$'s can also be interpreted as $\frac{1}{2}$-densities on Q and now we leave it to the reader to verify that if we interpret the $\dot{\psi}^\alpha$ as the local representations of $\frac{1}{2}$-densities on Q, then the results will be exactly the same as in section 7: the metalinear "correction" (with the metalinear bundle $\tilde{R}$ as given above) does not influence the quantization procedure in case of an oriented configuration space Q and $M = T^*Q$, equipped with the vertical polarization. The origin of the absence of any effect of the metalinear correction can be found in the fact that in example 3 of section 7 all (for this section interesting) absolute values were used on positive real numbers!

1D. Summary: $M = T^*Q$, $L = M \times \mathbb{C}$, $P = D^{v\mathbb{C}}$, Q oriented then

$$H = \{\text{square integrable } \tfrac{1}{2}\text{-densities on } Q\};$$

$f: M \to \mathbb{R}$ is quantizable iff $f = f_o + f_\xi$ where f_o is a function on Q and ξ a vector field on Q; for $\phi \in H$ we have

$$\delta(f_o + f_\xi)\phi = f_o\phi - i\hbar L_\xi \phi \ .$$

Moreover, if Q is an oriented Riemannian manifold, then the associated volume form ε enables us to identify $\frac{1}{2}$-densities with functions on Q and then:

$$H = L^2(Q,\varepsilon)$$

$$\delta(f_o + f_\xi) = f^o - \tfrac{1}{2}i\hbar\,\mathrm{div}(\xi) - i\hbar\xi$$

EXAMPLE 2: *the harmonic oscillator, holomorphic representation.*
In this example we investigate the modifications of example 1 of section 10. As these modifications are not as drastic (in a certain sense) as in the previous example, we only indicate the differences.

2B. P_{hol} and the bundles R, $\tilde{R}$ and $\tilde{B}^P$: P_{hol} is spanned globally by the hamiltonian vector field X_{p+iq}, so R is a trivial bundle:

$$R \cong \mathbb{R}^2 \times GL(1,\mathbb{C}) = \mathbb{R}^2 \times \mathbb{C}^*$$

with the identification $(q,p,\mu) \leftrightarrow (q,p,\mu\cdot X_{p+iq})$. The (unique) bundle $\tilde{R}$ is also a trivial bundle:

$$\tilde{R} \cong \mathbb{R}^2 \times ML(1,\mathbb{C}) = \mathbb{R}^2 \times \mathbb{C}^*$$

with the identification $\lambda \in \mathbb{C}^* \leftrightarrow (\lambda^2,\lambda) \in ML(1,\mathbb{C})$ and with projection:

$$\tilde{p}: \tilde{R} \to R, \quad (m,\lambda) \longmapsto (m,\lambda^2) = (m,\lambda^2 X_{p+iq}) .$$

The bundle $\tilde{B}^{P}{}^{hol}$ again is trivial since we have the global non-vanishing $-\frac{1}{2}$-P-form $\tilde{\nu}_o$ defined by

$$\tilde{\nu}_o(m,\lambda) = \frac{1}{\lambda} \in \mathbb{C}^* ;$$

moreover, this section is covariant constant along P because the section $\tilde{f}(m) = (m,1)$ of $\tilde{R}$ is a hamiltonian metaframe section of $\tilde{R}$ $(\tilde{p}\tilde{f}(m) = (m, X_{p+iq}))$. If we identify sections $\tilde{\nu}$ of $\tilde{B}^{P}{}^{hol}$ with functions $\dot{\nu}$ on M by means of $\tilde{\nu}_o$, then the partial connection on $\tilde{B}^{P}{}^{hol}$ becomes

$$(\nabla_\zeta \nu)^{\cdot} = \zeta\dot{\nu} ,$$

so the bundle QB is not altered in an essential way.

<u>2C. Quantization</u>: we follow example 1 of section 10, so we can identify H as

$$H = \{g: \mathbb{C} \to \mathbb{C} \mid g \text{ is holomorphic and}$$
$$\frac{1}{2\pi} \iint |g|^2 \exp\left(-\frac{p^2+q^2}{2\hbar}\right) dpdq < \infty\}$$

and an observable f is quantizable iff

$$f(q,p) = a\left(\frac{p^2+q^2}{2}\right) + bp + cq + d , \qquad a,b,c,d \in \mathbb{R} .$$

Because of the absence of the absolute values, $L_{X_H}\tilde{\nu}_o$ no longer vanishes and instead we have:

$$(L_{X_H}\tilde{\nu}_o)(m,\lambda) = \frac{d}{dt}\Big|_{t=0} \tilde{\nu}_o(\rho_t m, \tilde{\rho}_{t*}(\lambda^2 X_{p+iq}, \lambda))$$

$$= \frac{d}{dt}\Big|_{t=0} \tilde{\nu}_o(\rho_t m, (\lambda^2 e^{it} X_{p+iq}, e^{-it/2}\lambda))$$

$$= \frac{d}{dt}\Big|_{t=0} (\frac{1}{\lambda} e^{it/2}) = \frac{i}{2\lambda} = \tfrac{1}{2} i \tilde{\nu}_o(m,\lambda)$$

$$\Longleftrightarrow \quad L_{X_H}\tilde{\nu}_o = \frac{i}{2}\tilde{\nu}_o .$$

Consequently the action $\delta(H)$ on an element $g \in H$ is given by:

$$\delta(H)g(z) = \hbar\,(zg'(z) + \tfrac{1}{2}g(z))$$

and hence the $g_n(z) = z^n$ are eigenfunctions of $\delta(H)$ with eigenvalues $E_n = (n+\frac{1}{2})\hbar$, $n \in \mathbb{N}$, in accordance with quantum mechanics. Since $L_{X_p}\tilde{\nu}_o = L_{X_q}\tilde{\nu}_o = 0$, the operators $\delta(p)$ and $\delta(q)$ remain the same.

<u>2D. Summary</u>: $M = T^*\mathbb{R}$, $L = M \times \mathbb{C}$, $P = P_{hol}$ $\Rightarrow$

$$H = \{g: \mathbb{C} \to \mathbb{C} \mid \iint |g|^2 \exp(-|z|^2/2\hbar)\,dpdq < \infty\},$$

$f: M \to \mathbb{R}$ quantizable iff $f(q,p) = a\left(\frac{p^2+q^2}{2}\right) + bp + cq + d.$

$$\delta(H) = \hbar(z\frac{d}{dz} + \tfrac{1}{2}) , \quad \delta(p) = \tfrac{1}{2}z + \hbar\frac{d}{dz}, \quad \delta(q) = \frac{-i}{2}z + i\hbar\frac{d}{dz}$$

$$\tfrac{1}{2}\delta(p)^2 + \tfrac{1}{2}\delta(q)^2 = \delta(\tfrac{1}{2}p^2 + \tfrac{1}{2}q^2).$$

<u>EXAMPLE 3</u>: *harmonic oscillator, energy representation.*

In this example we reconsider example 5 of section 7. Two possible choises for the metalinear bundle $\tilde{R}$ are available: one of them (the trivial one) will not influence the results, the other will.

<u>3b. P_{en} and the bundles R, $\tilde{R}$ and $\tilde{B}^{P_{en}}$</u>: the vector field $X_\rho = \frac{\partial}{\partial\phi}$ is a global non-vanishing hamiltonian vector field which spans P_{en} hence R is trivial:

$$R \cong M \times \mathbb{C}^* \ , \quad (m,\mu) = \left(m, \mu \frac{\partial}{\partial\phi}\right)$$

A possible choice for $\tilde{R}$ is given by $\tilde{R}_{triv}$:

$$R_{triv} = M \times \mathbb{C}^* , \quad p(m,\lambda) = (m,\lambda^2) = \left(m,\lambda^2 \frac{\partial}{\partial\phi}\right) ;$$

and a global trivializing section $\tilde{\nu}_o$ of $\tilde{B}^{P}{}_{en}$ is given by:

$$\tilde{\nu}_o(m,\lambda) = \lambda^{-1} .$$

Since the section $f(m) = (m,1)$ is a global hamiltonian metaframe, it follows that $\tilde{\nu}_o$ is covariant constant along P , so if $\tilde{\nu} = \hat{\nu} \cdot \tilde{\nu}$ is a $-\frac{1}{2}$-P-density then:

$$(\nabla_\zeta \tilde{\nu})^\cdot = \zeta \hat{\nu}$$

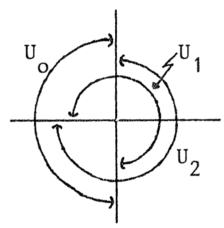
and the quantization as given in example 5 of section 7 remains the same.

However, another choice for $\tilde{R}$ is possible; therefore consider the open cover of M by U_o, U_1, U_2 defined by:

$$U_o = \{(\rho,\phi) \in M \mid \rho \in \mathbb{R}^+, \tfrac{1}{2}\pi < \phi < \tfrac{3}{2}\pi\}$$

$$U_1 = \{(\rho,\phi) \in M \mid \rho \in \mathbb{R}^+, -\tfrac{1}{2}\pi < \phi < \pi\}$$

$$U_2 = \{(\rho,\phi) \in M \mid \rho \in \mathbb{R}^+, -\pi < \phi < \tfrac{1}{2}\pi\}$$



In these charts, the transition functions of the bundle R are given by $g_{01} = g_{12} = g_{02} = 1 = \mathrm{id} \in GL(1,\mathbb{C}) \cong \mathbb{C}^*$. Since there is no triple intersection, we can choose the $\tilde{g}_{ij}$ freely, and we now choose:

$$\tilde{g}_{01} = \tilde{g}_{12} = 1 = (1,1) \in ML(1,\mathbb{C}) \cong \mathbb{C}^*$$

$$\tilde{g}_{02} = -1 = (1,-1) \in ML(1,\mathbb{C}) \cong \mathbb{C}^*$$

[N.B. the first choice of $\tilde{g}_{ij}$ for $\tilde{R}_{triv}$ corresponds to $\tilde{g}_{ij} = 1$]. The transformation functions of the bundle $\tilde{B}^{P}{}_{en}$ now become

$$\mu_{01} = \mu_{12} = 1 \ , \quad \mu_{02} = -1$$

and the complex line-bundle $\tilde{B}^{P_{en}}$ admits a global non-vanishing section $\tilde{\nu}_o$ defined by

$$U_i \times ML(1,\mathbb{C}) \ni (m,\lambda) \xrightarrow{\tilde{p}} \left(m, \lambda^2 \frac{\partial}{\partial\phi}\right)$$

$$\tilde{\nu}_o\big|_{U_i}(m,\lambda) = \lambda^{-1} \cdot \exp(i\phi(m)/2) \quad \text{if} \quad m = (\rho,\phi) .$$

If we now compute the covariant derivative of $\tilde{\nu}_o$ along $X_\rho = \frac{\partial}{\partial\phi}$ then we get:

$$(\nabla_{X_\rho}\tilde{\nu}_o)\big|_{U_i}(m,1) = \frac{\partial}{\partial\phi}\tilde{\nu}_o(m,1) = \frac{\partial}{\partial\phi}(\exp(i\phi/2)) = \frac{i}{2}\tilde{\nu}_o(m,1)$$

this is a hamiltonian metaframe!

hence $\nabla_{X_\rho}\tilde{\nu}_o = \frac{i}{2}\tilde{\nu}_o$.

<u>3C. Quantization</u>: when we compute the condition that ψ (a section of QB) should be covariant constant along P_{en} , then:

$$X_H\dot{\psi} - \frac{i}{\hbar}\rho\dot{\psi} + \frac{i}{2}\dot{\psi} = 0$$

$$\Longleftrightarrow \quad \dot{\psi}(\rho,\phi) = h(\rho)\exp(i(\frac{\rho}{\hbar} - \tfrac{1}{2})\phi)$$

hence the allowed values of ρ are given by

$$\rho = (n+\tfrac{1}{2})\hbar \ , \quad n \in \mathbb{N}$$

<u>3D. Summary and conclusions</u>: if we adopt the viewpoint of example 5 of section 7 and if we use the second metalinear bundle $\tilde{R}$, then we identify the Hilbert space $\mathcal{H}$ with square summable series:

$$\mathcal{H} = \left\{(a_n)_{n\in\mathbb{N}} \ \middle| \ \sum_{n=o}^{\infty} |a_n|^2 < \infty, \ a_n \in \mathbb{C}\right\}$$

$f: M \to \mathbb{R}$ is quantizable iff $f(\rho,\phi) = f(\rho)$ and then:

$$\delta(f)\left\{a_n\right\}_{n\in\mathbb{N}} = \left\{f((n+\tfrac{1}{2})\hbar)\,a_n\right\}_{n\in\mathbb{N}} .$$

<u>EXAMPLE 4</u>: *the free moving particle on* S^2 .

In this final example we reconsider example 2 of section 10; again only the differences due to the metalinear correction will be given.

<u>4B. P_{en} and the bundles R, $\tilde{R}$ and $\tilde{B}^{P_{en}}$</u>: the vector fields X_H and V are globally non-vanishing, independent and they span P_{en} hence R is a trivial bundle:

$$R \cong M \times GL(2,\mathbb{C})$$

$$(m,g) \cong (m, (X_H,V)g) ,$$

and we obtain a global non-vanishing section $\tilde{\nu}_o$ of $\tilde{B}^{P_{en}}$ by

$$\tilde{\nu}_o(m, \widetilde{id}) = 1 .$$

<u>4C. Quantization</u>: we have to compute the covariant derivative $\nabla_\zeta\tilde{\nu}_o$ for $\zeta \in P_{en}$ in order to obtain the condition on sections of QB to be covariant constant along P_{en}.

$$(\nabla_\zeta\tilde{\nu}_o)(m_o, \widetilde{id}) = \zeta_{m_o}\,\tilde{\nu}_o(m, \tilde{g}(m))$$

where $\tilde{g}(m) \in ML(2,\mathbb{C})$ is such that $(X_H,V)\cdot p(\tilde{g}(m))$ is a hamiltonian frame and such that $\tilde{g}(m_o) = \widetilde{id}$. Now observe that there exists (at least locally) a complex function w (w is in fact a holomorphic coordinate on M/D) such that

$$X_w(m) = \lambda(m)V(m)$$

for some function $\lambda(m)$ [see example 2 of section 10 for an explicit expression for w], hence if $p(\tilde{g}(m)) = g(m)$ is given by

$$g(m) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\lambda(m)}{\lambda(m_o)} \end{pmatrix} \qquad [\Rightarrow g(m_o) = \mathrm{id}]$$

then $(X_H, V) \cdot g(m) = \left(X_H, \frac{X_w}{\lambda(m_o)}\right)$ is hamiltonian, hence

$$\tilde{\nu}_o(m, g(m)) = \sqrt{\lambda(m_o)/\lambda(m)}$$

where we have to use that branch of $\sqrt{\ }$ (in a neighbourhood of m_o) such that $\sqrt{1} = 1$. With these definitions we get:

$$(\nabla_\zeta \tilde{\nu}_o)(m_o, \tilde{id}) = \frac{-1}{2\lambda(m_o)}\left(\zeta_{m_o} \lambda(m)\right)\tilde{\nu}_o(m_o, \tilde{id}).$$

A lengthy calculation yields

$$X_H\big|_{m_o} \lambda(m) = -iv\,\lambda(m_o) \quad \text{and} \quad V\big|_{m_o} \lambda(m) = 0$$

so: $$\nabla_{X_H}\tilde{\nu}_o = \frac{iv}{2}\tilde{\nu}_o \quad \text{and} \quad \nabla_V \tilde{\nu}_o = 0.$$

Using these results, a section $\psi = \dot{\psi}_a\, s_o \otimes \nu_o$ is covariant constant along P_{en} iff (on the chart U_a):

$$\begin{cases} X_H\dot{\psi}_a - \frac{i}{\hbar}\theta(X_H)\dot{\psi}_a + \frac{iv}{2}\dot{\psi}_a = 0 \\ V\dot{\psi}_a - \frac{i}{\hbar}\theta(V)\dot{\psi}_a = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} v\,\frac{\partial\dot{\psi}_a}{\partial t_a} = \frac{i}{\hbar}v^2\dot{\psi}_a - \frac{iv}{2}\dot{\psi}_a = iv\left(\frac{v}{\hbar} - \frac{1}{2}\right)\dot{\psi}_a \\ i\left(\frac{\vec{y}}{v} + i\vec{x}\right)\frac{\partial\dot{\psi}_a}{\partial\vec{n}} + \frac{n_a}{1-n_a^2}\left(\frac{y_a}{v} + ix_a\right)\frac{\partial\dot{\psi}_a}{\partial t_a} = 0 \end{cases}$$

hence one solution is given by

$$\dot{\psi}_a(\vec{n}, v, t_a) = h_a(v)\left(1-n_a^2\right)^{\frac{1}{2}(\frac{v}{\hbar}-\frac{1}{2})} \exp(i(\frac{v}{\hbar}-\frac{1}{2})t_a)$$

and the condition that t_a is a cyclic coordinate implies:

$$(h_a(v) \neq 0 \Rightarrow \frac{v}{\hbar}-\tfrac{1}{2} \in \mathbb{Z}) \Rightarrow v = (\ell+\tfrac{1}{2})\hbar , \quad \ell \in \mathbb{N} .$$

We now introduce the variable $\lambda = \frac{v}{\hbar} - \frac{1}{2}$ and we remind ourselves that λ is restricted to the natural numbers: $\lambda \in \mathbb{N}$ (N.B. this is the same trick we used in example 2 of section 10. If we choose $h_a(\lambda) = ((\lambda+\frac{1}{2})\hbar)^{-1}$ ($\Longleftrightarrow h_a(v) = v^{-1}$ for the allowed values of v) then:

$$\dot{\psi}_a(\vec{n}, \lambda, t_a) = \frac{1}{(\lambda+\frac{1}{2})\hbar}(1-n_a^2)^{\frac{1}{2}} \exp(i\lambda t_a)$$

$$<\psi_a,\psi_a> = \sum_{\lambda=o}^{\infty} \int_{S^2} d\Omega\ (1-n_a^2)^{\lambda} .$$

If we compare these results with the previous ones, we see that the interpretation of the Hilbert space H does not change:

$$H = \left\{\psi \mid \psi = \left(\sum_{m=-\lambda}^{\lambda} a_{\lambda m}\, w_1^m\right)\psi_1 , <\psi,\psi> < \infty\right\} .$$

The observables H and L_i remain quantizable and:

$$\delta(H)\psi = \tfrac{1}{2}v^2\psi = \tfrac{1}{2}\hbar^2(\lambda+\tfrac{1}{2})^2\psi$$

hence the $\psi_{\ell m}$ are eigenfunctions of $\delta(H)$ with corresponding eigenvalues $E_\ell = \frac{1}{2}\hbar^2(\ell+\frac{1}{2})^2$.

Since $[X_{L_i},X_H] = [X_{L_i},V] = 0$ it follows that

$$\begin{aligned}\left(L_{X_{L_i}}\tilde{\nu}_o\right)(m, \tilde{id}) &= \frac{d}{dt}\Big|_{t=0} \tilde{\nu}_o(\rho_t m, \tilde{\rho}_{t*}\tilde{id}) \\ &= \frac{d}{dt}\Big|_{t=0} \tilde{\nu}_o(\rho_t m, \tilde{id}) = 0\end{aligned}$$

$\Big($because $p(\widetilde{\rho}_{t*}\,\widetilde{id}) = \rho_{t*}\Big(X_H\big|_m, V\big|_m\Big) = \Big(X_H\big|_{\rho_t m}, V\big|_{\rho_t m}\Big)\Big)$ so $\delta(L_i)\dot{\psi}\ s_o \otimes \widetilde{\nu}_o =$ $= -i\hbar(X_{L_i}\dot{\psi})\ s_o \otimes \widetilde{\nu}_o$ and the action of $\delta(L_i)$ upon the $\psi_{\ell m}$ remains the same.

<u>4D. Summary and conclusions:</u> $M = T^*S^2 \setminus \{0\}$, $L = M \times \mathbb{C}$, $P = P_{en}$ and the trivial metalinear framebundle imply (using our heuristic approach to the inner product):

$$H = \left\{\psi : M \to QB \mid \psi = \left(\sum_{m=-\lambda}^{\lambda} a_{\lambda m}\, w_1^m\right)\psi_1 \wedge <\psi,\psi> < \infty\right\}$$

$$<\psi,\hat{\psi}> = \sum_{\lambda=o}^{\infty} \int_{S^2} d\Omega \sum_{m,m'=-\lambda}^{\lambda} \bar{a}_{\lambda m}\, \hat{a}_{\lambda m'}\, \bar{w}_1^m\, w_1^{m'}\,(1-n_1^2)^{\lambda}\ .$$

The observables $H = \frac{1}{2}v^2$ and $L = x \wedge y$ are quantizable; the eigenvalues E_ℓ of $\delta(H)$ are given by $E_\ell = \frac{1}{2}\hbar^2(\ell+\frac{1}{2})^2$ with corresponding eigenfunctions $\psi_{\ell m}$ ($\Longleftrightarrow a_{\lambda m'} = \delta_{\lambda\ell}\cdot\delta_{mm'}$). The operator $\frac{1}{2}(\delta(L_1)^2 + \delta(L_2)^2 + \delta(L_3)^2)$ has the same eigenfunctions $\psi_{\ell m}$ as $\delta(H)$, but different eigenvalues F_ℓ given by $F_\ell = \frac{1}{2}\hbar^2\ell(\ell+1)$ hence:

$$\delta\left(\tfrac{1}{2}L_1^2 + \tfrac{1}{2}L_2^2 + \tfrac{1}{2}L_3^2\right) = \tfrac{1}{2}\left(\delta(L_1)^2 + \delta(L_2)^2 + \delta(L_3)^2\right) + \tfrac{1}{8}\hbar^2 .$$

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